Lattice Fermion Anomalies & CPT

arXiv:2508.17115 w/ Seiberg&Shao
arXiv:2601.01191 w/ Seiberg

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Princeton University

New Developments in Lattice Fermions, Syracuse University

Goals of the talk

- A systematic approach to probe lattice anomalies and match to the continuum using group theory.

- Two examples: order 8 anomalies on the lattice (e.g. parity anomaly in 2+1d).

- Demonstrate how lattice CPT operator helps to match the lattice anomaly and continuum anomaly.

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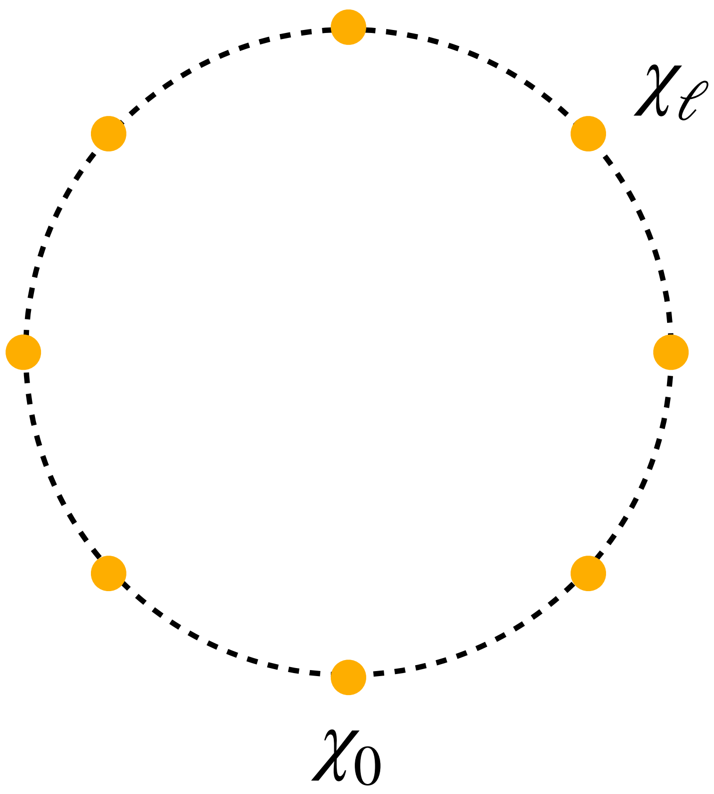
- Order: the smallest # of copies s.t. anomaly-free.

An example: 1+1d Majorana chain



L sites, one Majorana fermion per site

$$\chi_\ell = \chi_{\ell+L}, \quad \{\chi_\ell, \chi_{\ell'}\} = 2\delta_{\ell,\ell'}$$

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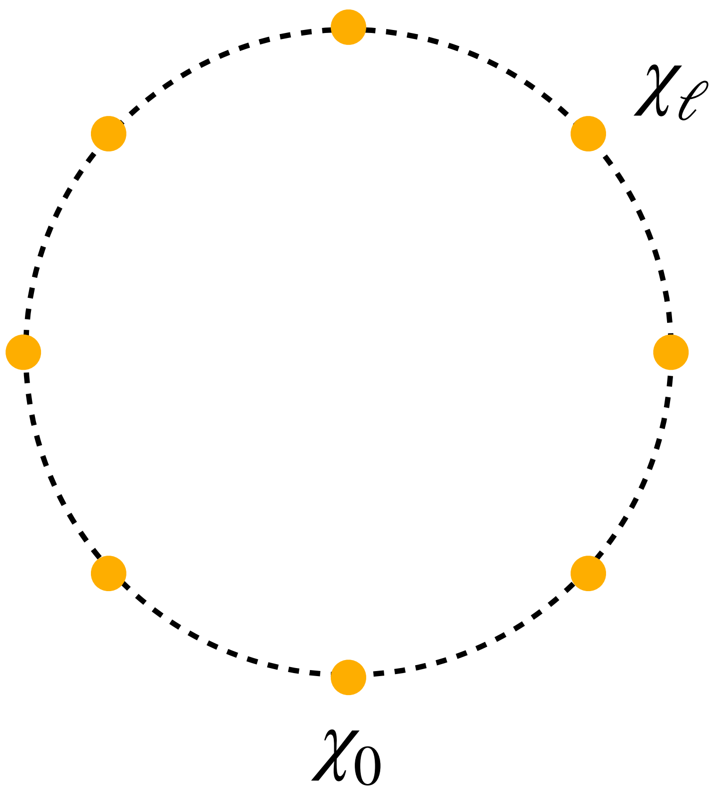

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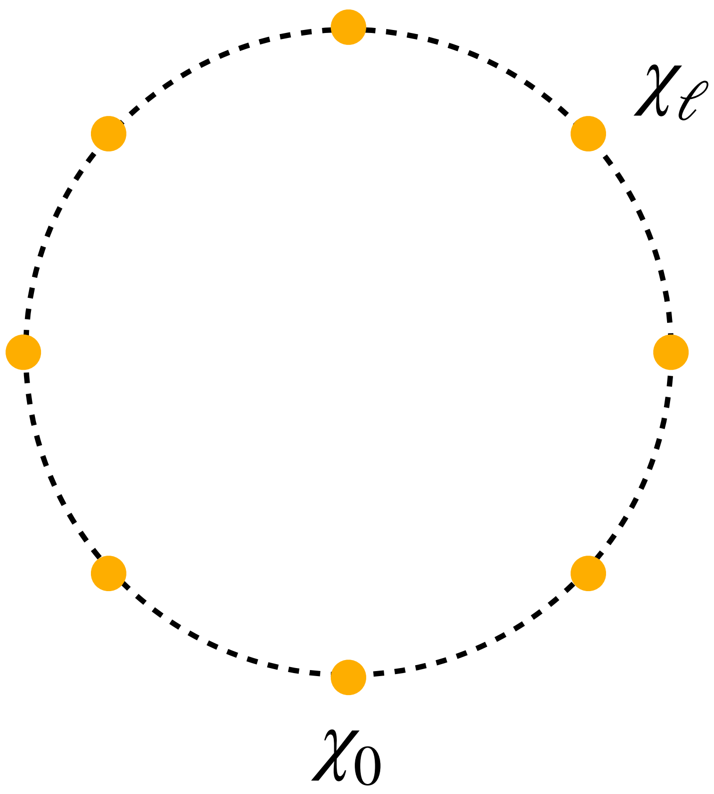

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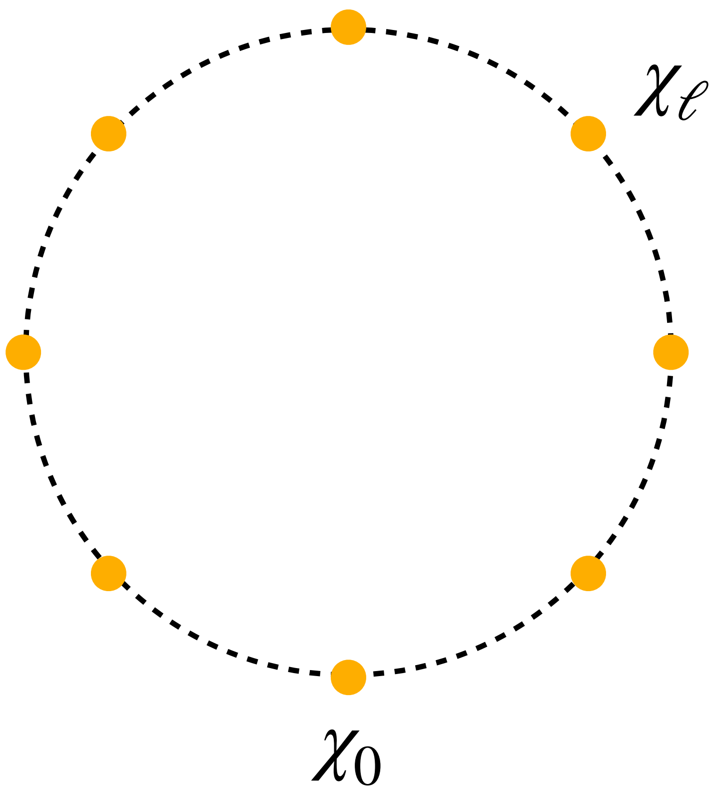

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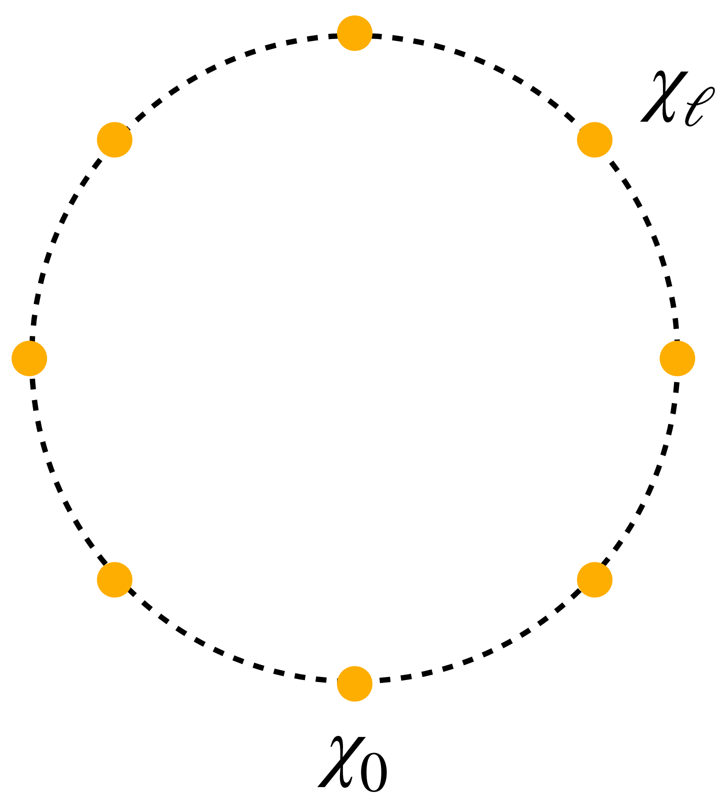

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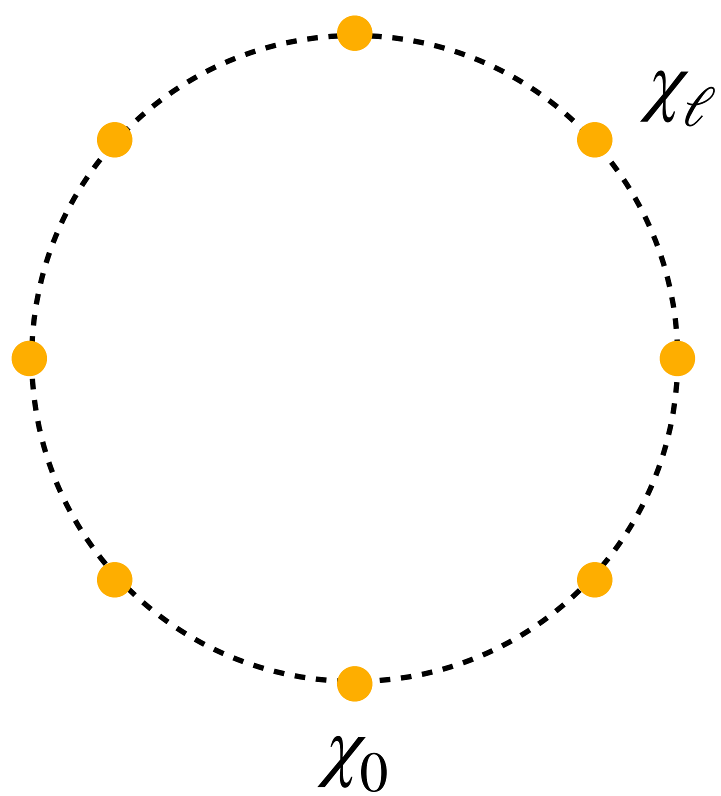

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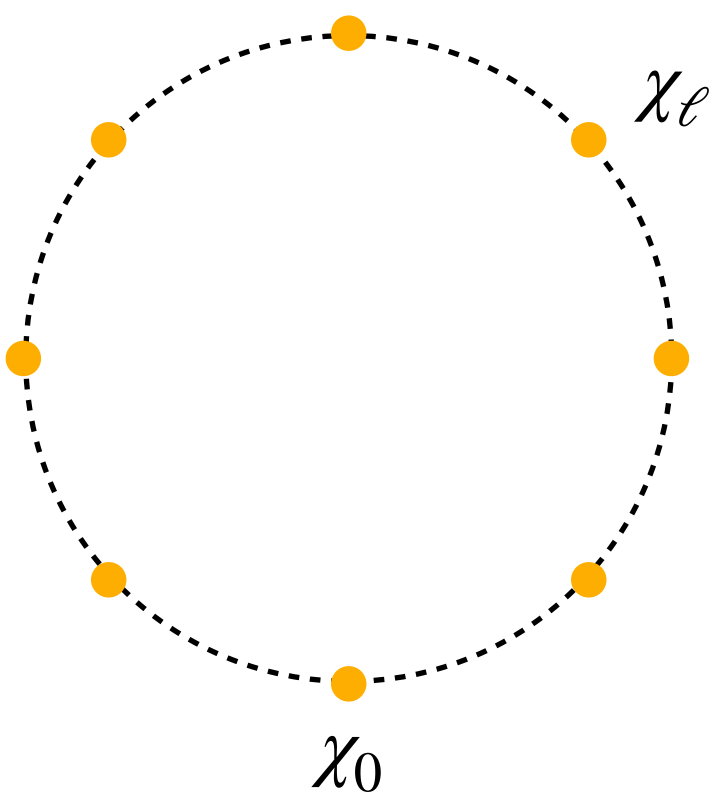

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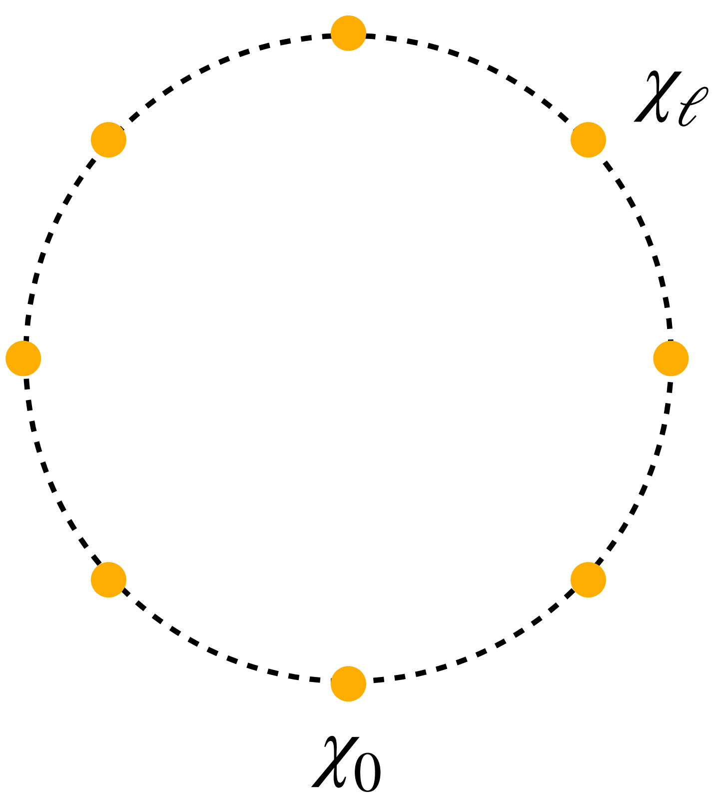

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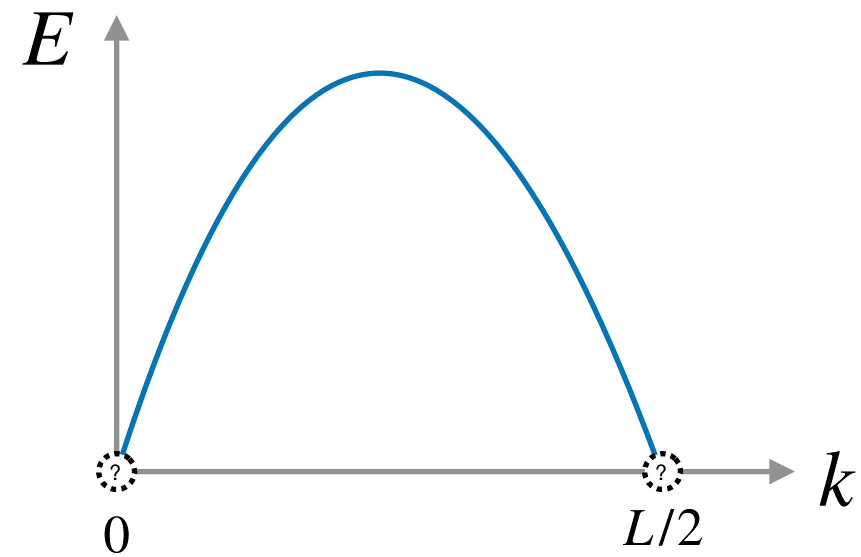

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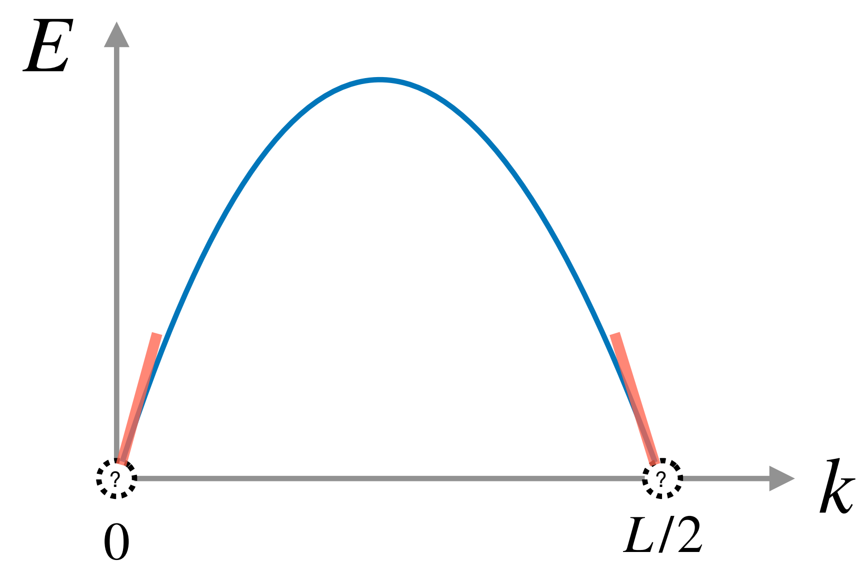

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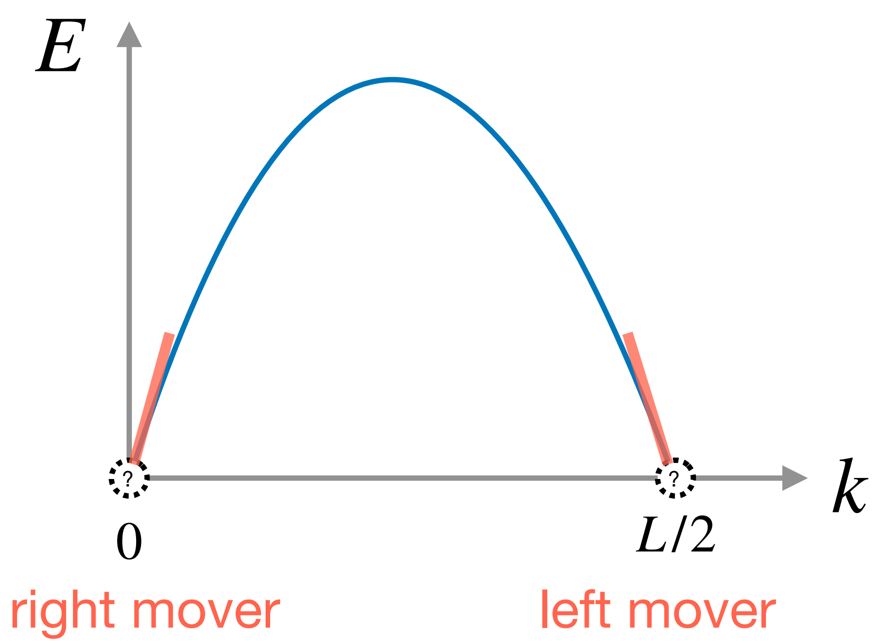

The lattice translation leads to internal symmetry in the continuum.

$$T \to \mathsf{C} \qquad \mathsf{C}\chi_{\rm L}\mathsf{C}^{-1} = -\chi_{\rm L}, \quad \mathsf{C}\chi_{\rm R}\mathsf{C}^{-1} = \chi_{\rm R}$$

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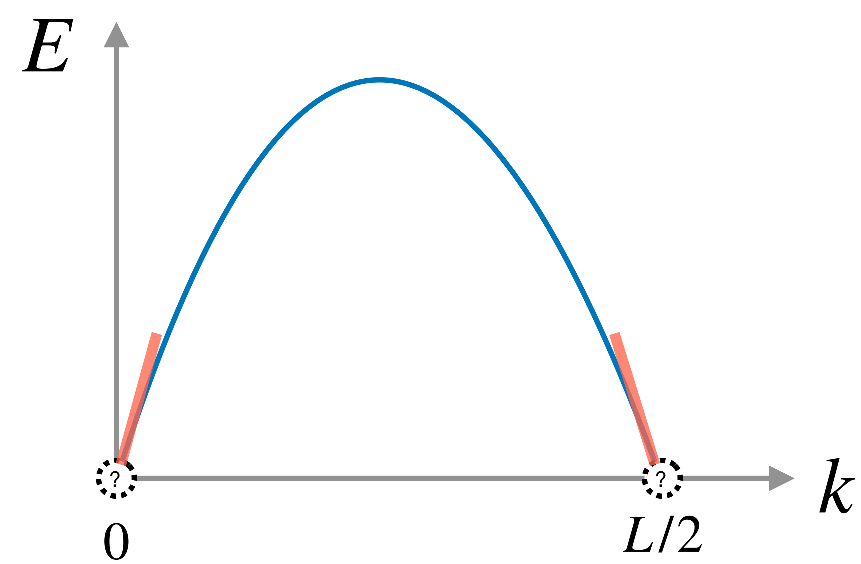

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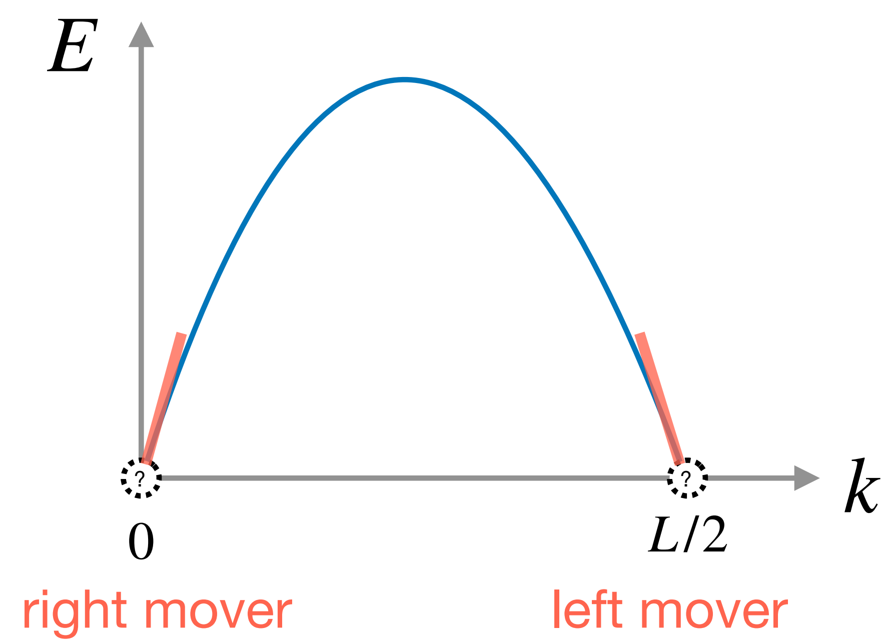

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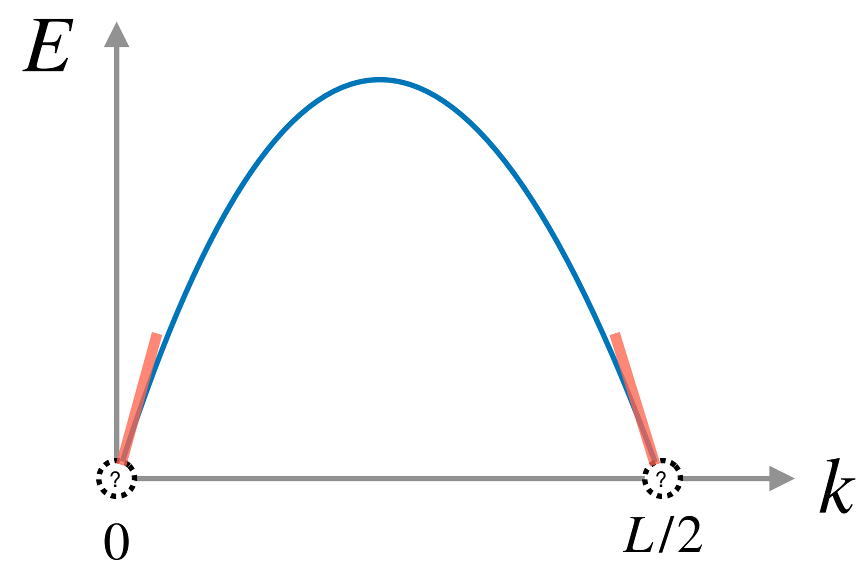

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The lattice anomaly is mapped to the continuum anomaly (L even).

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The continuum anomaly of the emanant $\mathbb{Z}_2$ symmetry is emergent.

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(See also [Lew-Smith, Pace, Shao] and Sal's talk)

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- In Lorentzian signature, $\Theta^2 = 1$.
- For any internal symmetry U of the theory, $\Theta \mathrm{U} = \mathrm{U} \Theta$.
- In odd space dimensions, one can combine Θ with all other spatial reflections (equivalent to product of π spatial rotation), which produces CPT.

The anomaly involving CRT

Lattice $\Pi \rightarrow \Theta$ flows to the continuum CRT.

$$\Theta \chi_{\mathrm{L}}^{A}(t, x) \Theta^{-1} = -\chi_{\mathrm{L}}^{A}(-t, -x), \quad \Theta \chi_{\mathrm{R}}^{A}(t, x) \Theta^{-1} = \chi_{\mathrm{R}}^{A}(-t, -x)$$

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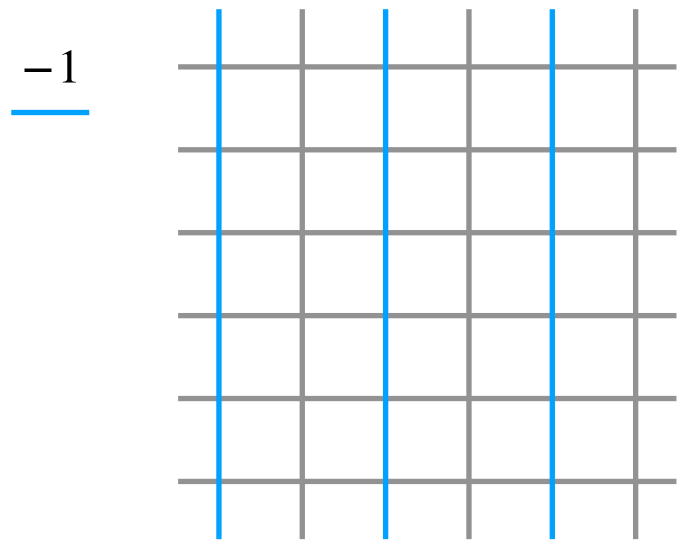
- One Majorana fermion per site, $\mathbb{Z}_2$ background gauge field with flux -1 on each plaquette [Kogut, Susskind; Banks, Susskind, Kogut].



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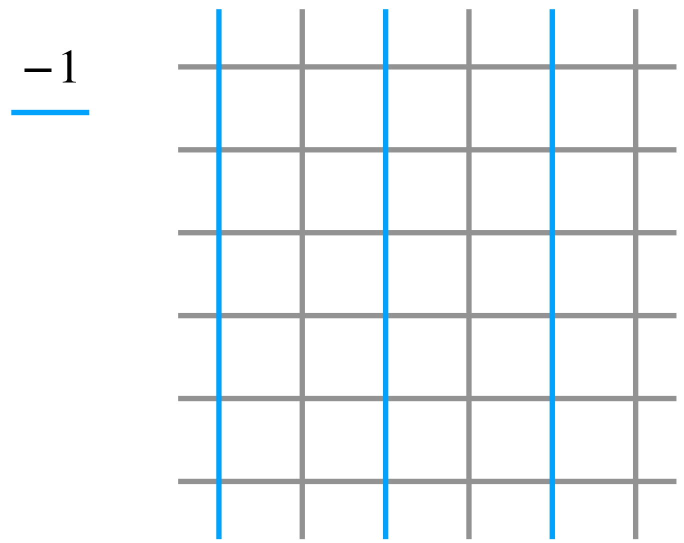

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$$\eta_\mu(\vec{\ell}) = (-1)^{\sum_{\nu<\mu} \ell^\nu} = \begin{cases} 1 & \mu = 1 \\ (-1)^{\ell_1} & \mu = 2 \end{cases}$$

- Symmetries [Kluberg-Stern, Morel, Napoly, Petersson; Parisi, Zhang; Golterman, Smit; Golterman; Golterman; Kilcup, Sharpe]:

$$\mathscr{C}^4 = 1, \quad \mathscr{C} T_1 \mathscr{C}^{-1} = T_2, \quad \mathscr{C} T_2 \mathscr{C}^{-1} = (-1)^F T_1^{-1}, \quad T_1 T_2 = (-1)^F T_2 T_1,$$
$$\mathscr{R}^2 = 1, \quad T_1 \mathscr{R} = (-1)^F \mathscr{R} T_1^{-1}, \quad \mathscr{R} \mathscr{C} \mathscr{R}^{-1} = \mathscr{C}^{-1},$$
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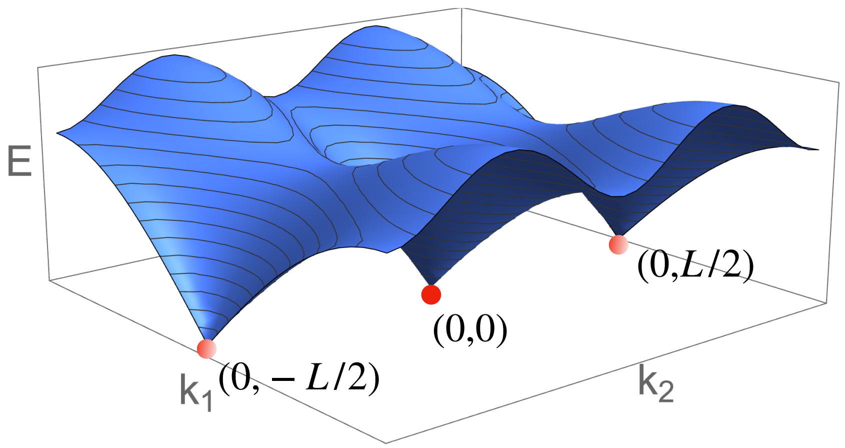
2+1d Staggered Fermion



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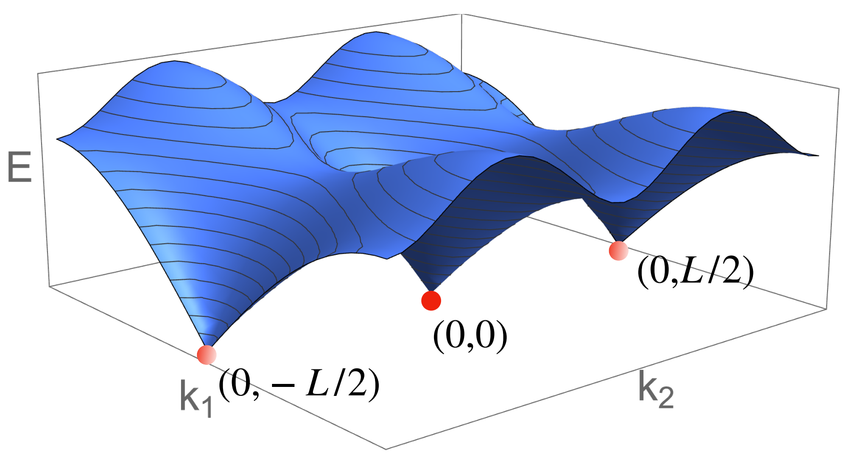

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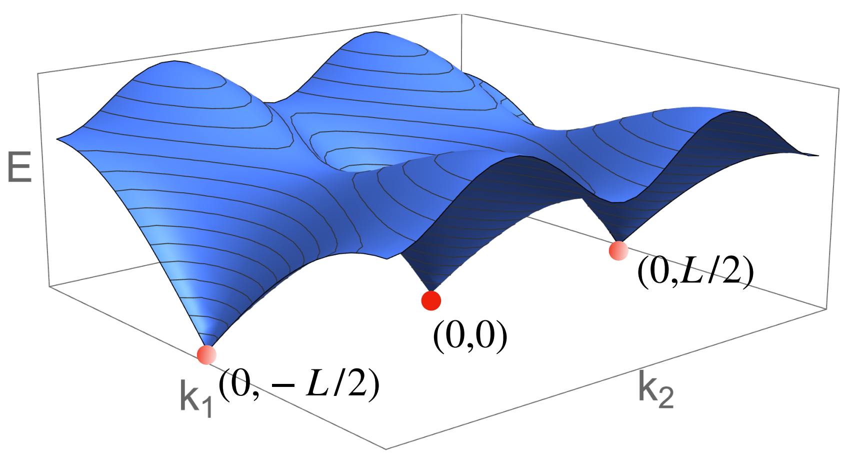

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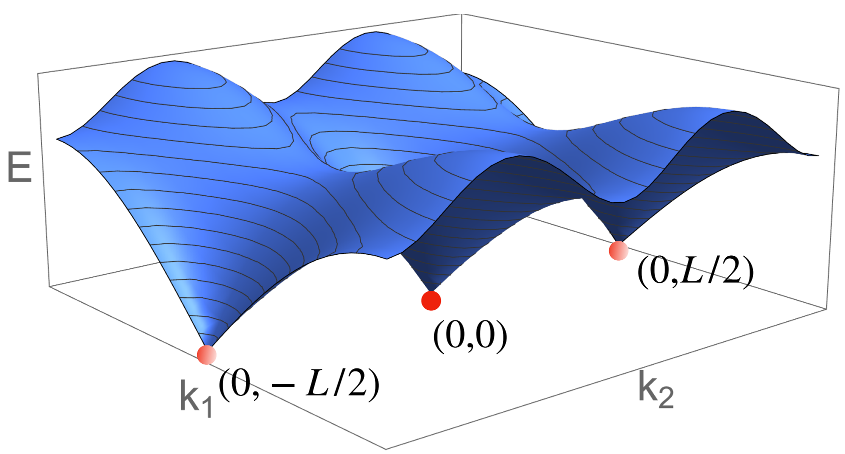

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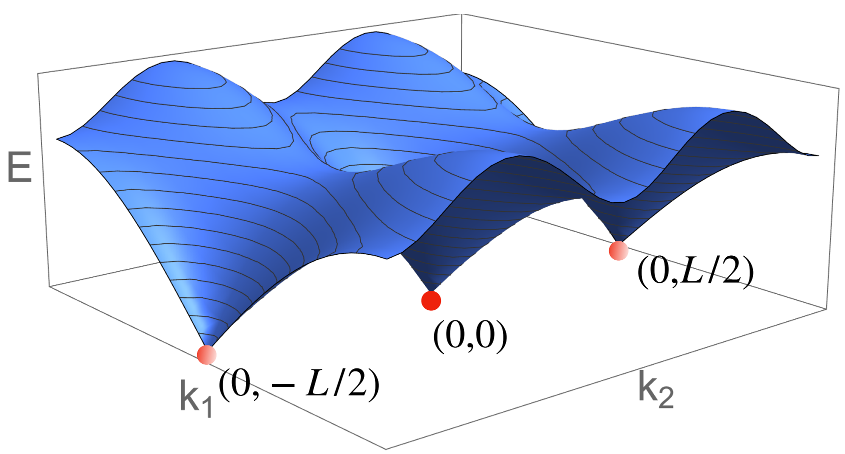

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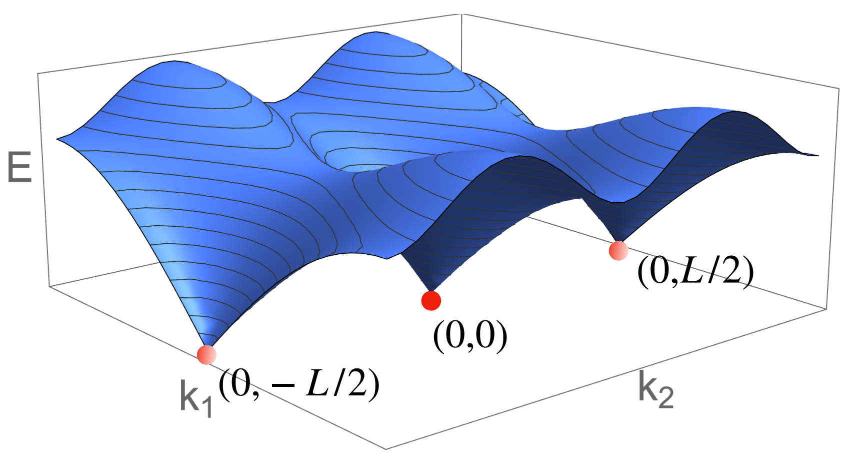

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$$\langle \Gamma_1, \Gamma_2 \rangle = D_4 \subset O(2),$$

$$\Gamma_1^2 = \Gamma_2^2 = 1, \quad \Gamma_1\Gamma_2 = (-1)^F\Gamma_2\Gamma_1$$

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- These methods involve non-compact/non-flat space or operations that are hard to realize on the lattice.

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Related work:

- Parity anomaly in the tight-binding model [Pace, Kim, Chatterjee, Shao]

- Staggered fermion w/ gravitational background [Catterall]

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We need a systematic approach to handle these data.

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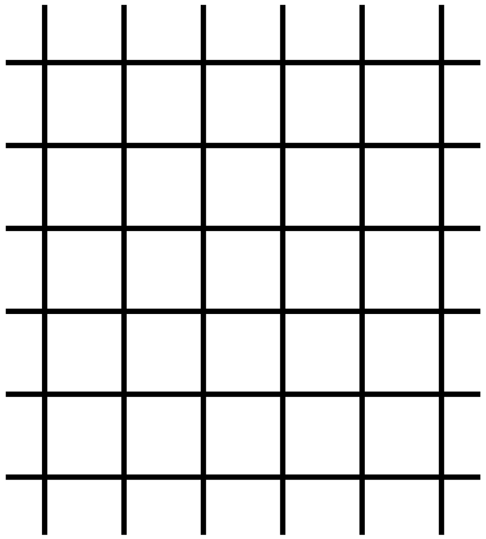
Start with infinite lattice $\mathbb{Z}^2$, local degrees of freedom $\psi_{\vec{\ell}}$ and Hamiltonian H. The symmetry is group G.



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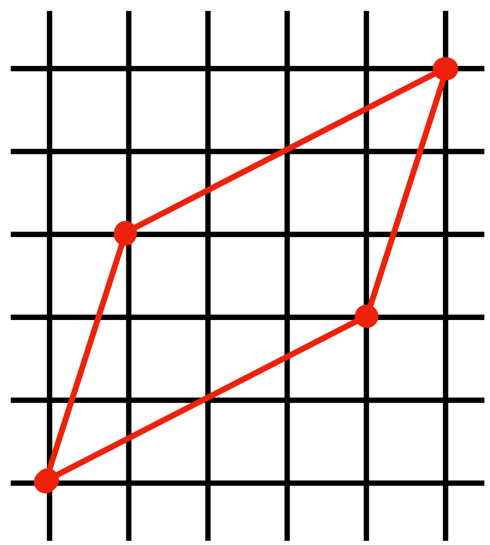

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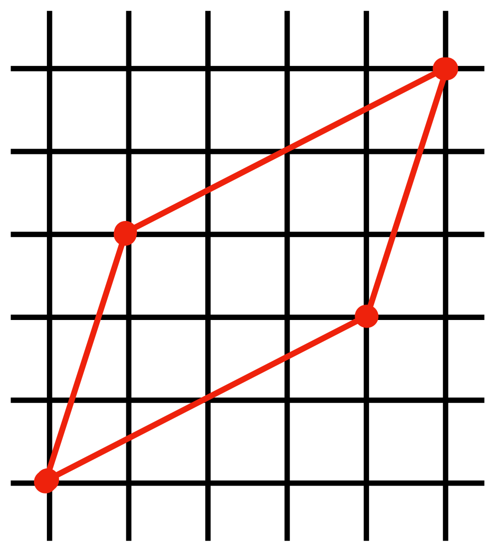

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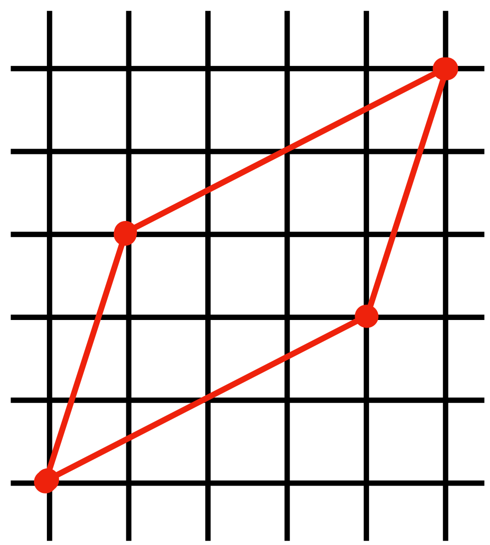

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The group $\mathcal{G} = \langle g_1, g_2 \rangle$ must be the fundamental group of the resulting manifold. And it gives the full list of boundary conditions.

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When restricting to the internal symmetries (no change to the coordinate), it becomes the centralizer,

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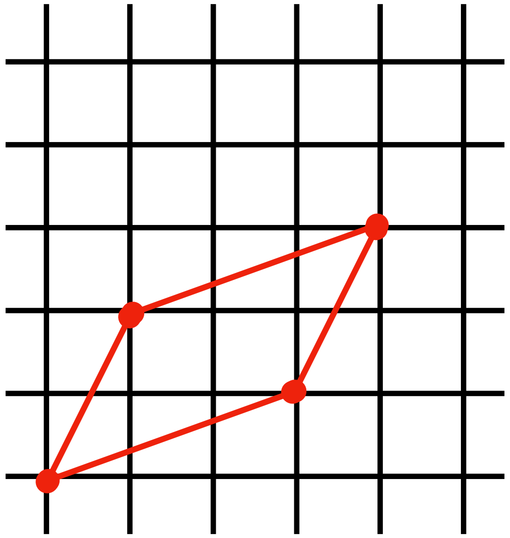

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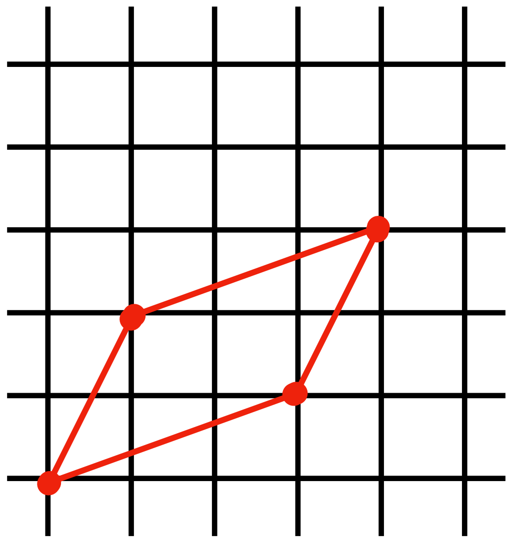

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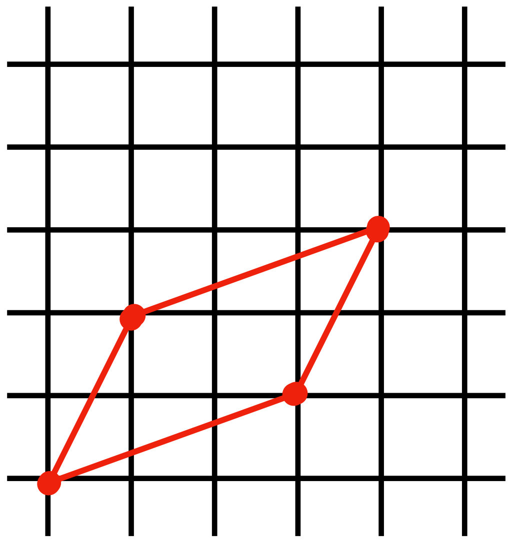

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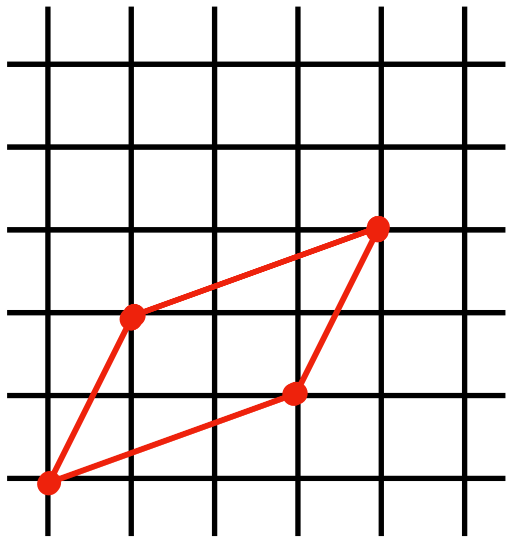

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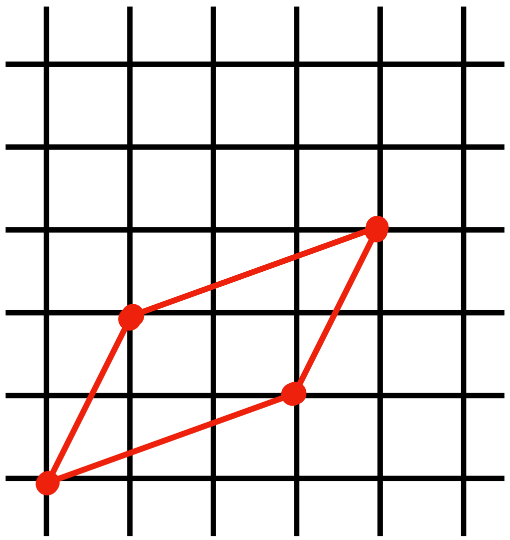

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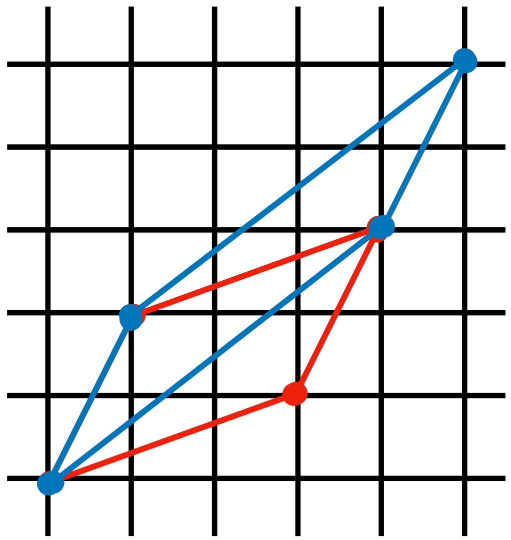

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- Geometrically, this is (universal) group $\mathscr{K}$-covering and the symmetry becomes the isometry of the manifold.

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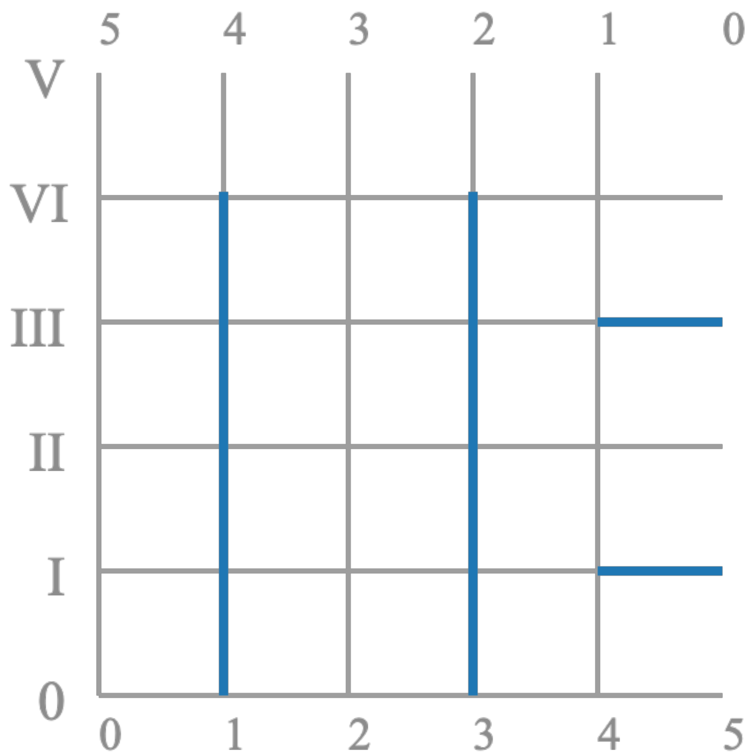
- The projective phases of these symmetries can be analyzed on the zero modes and hence easy to match as well.

Example: Order 8 (# copies N_f even)



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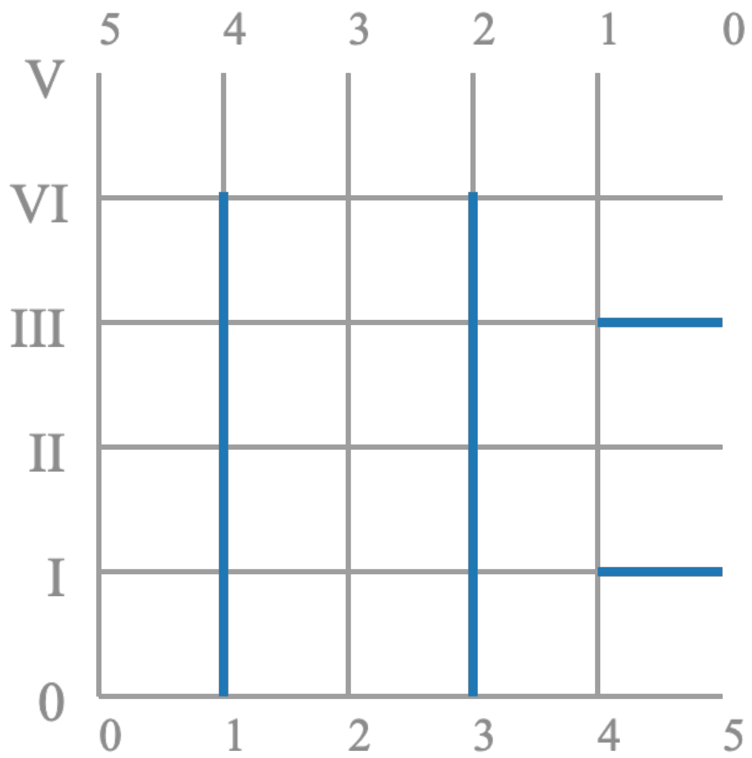

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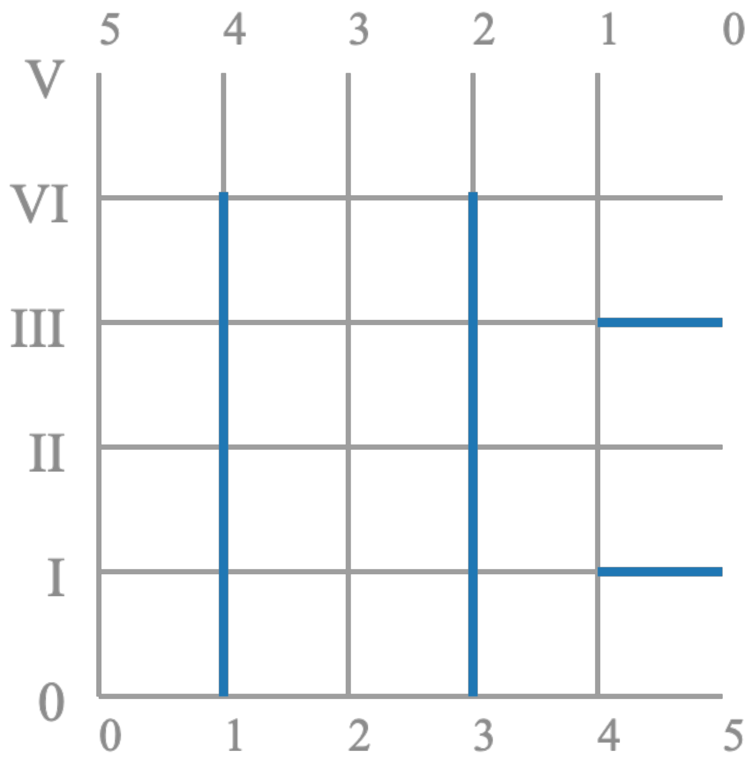

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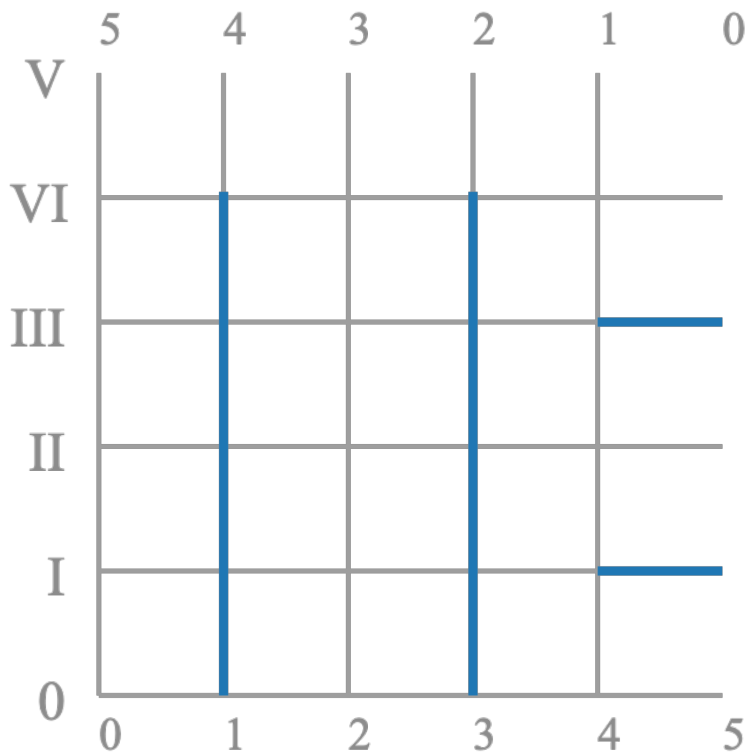

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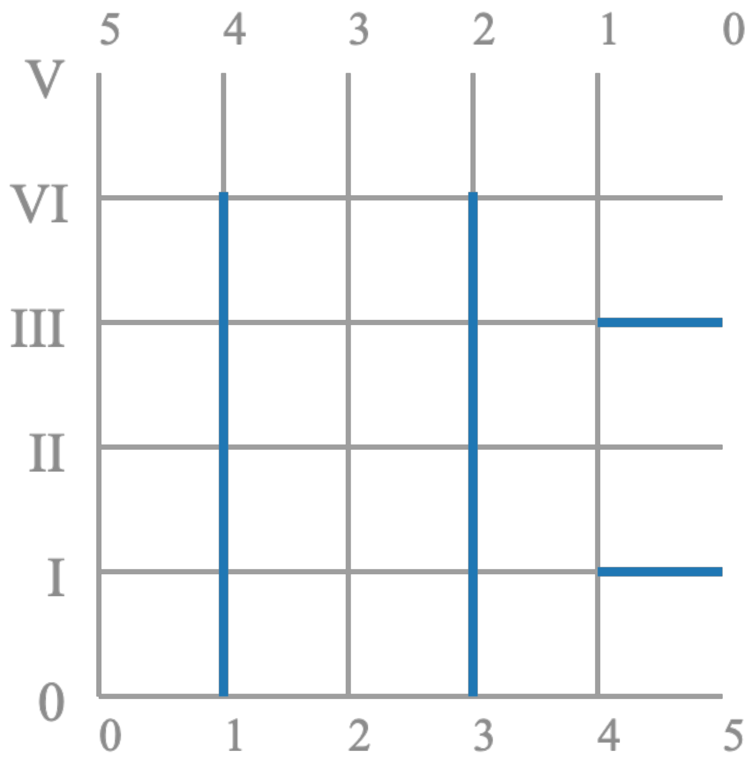

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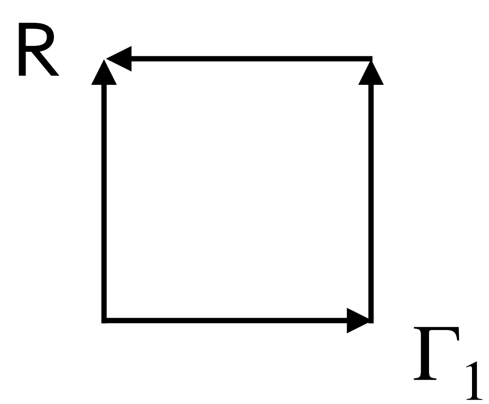

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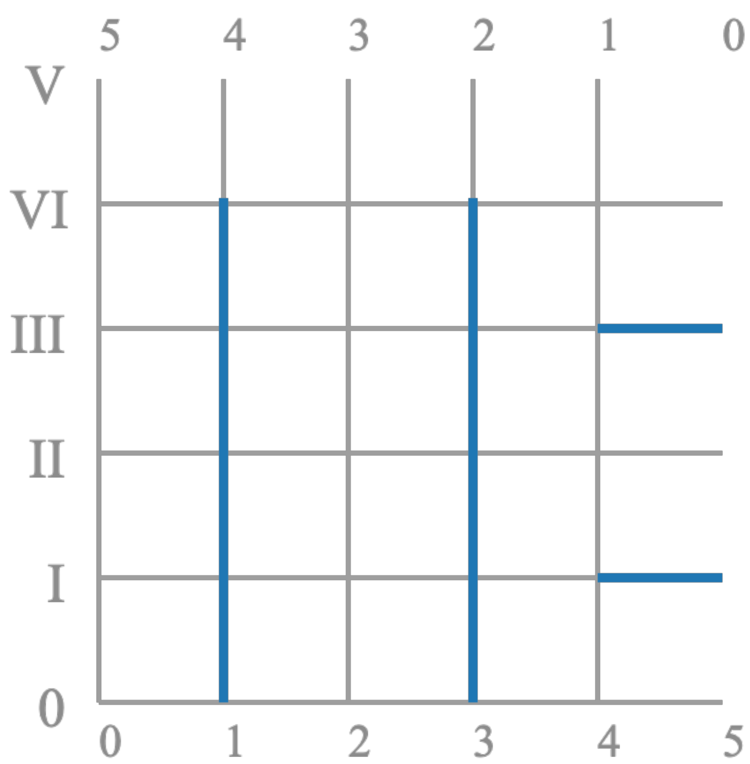

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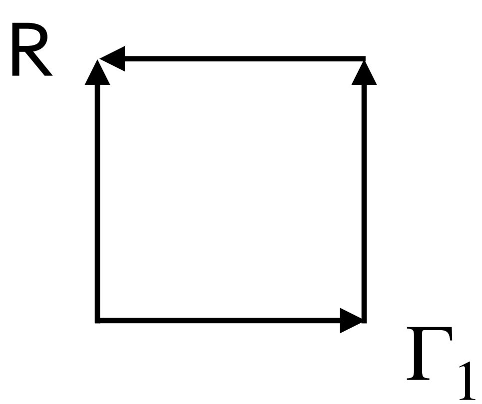

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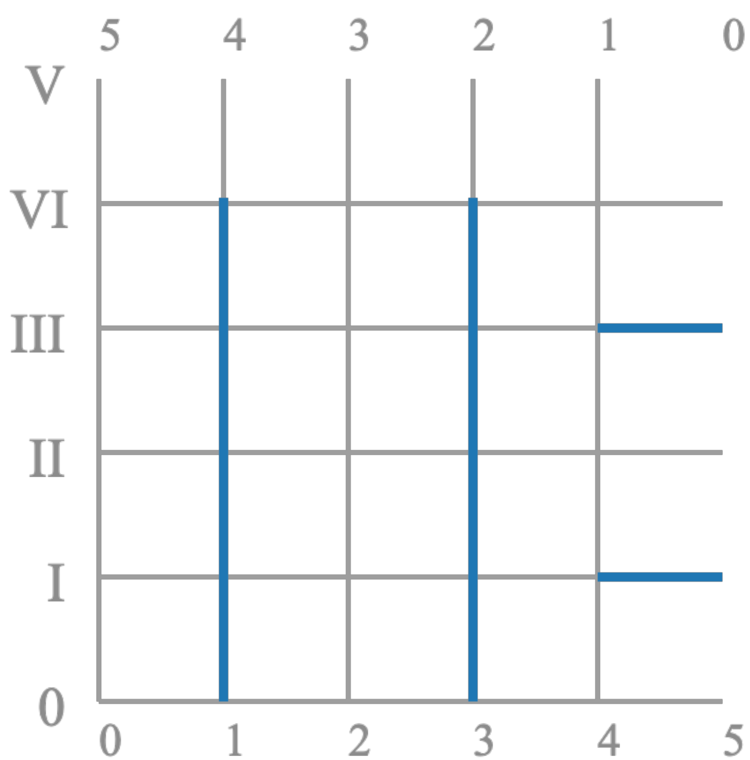

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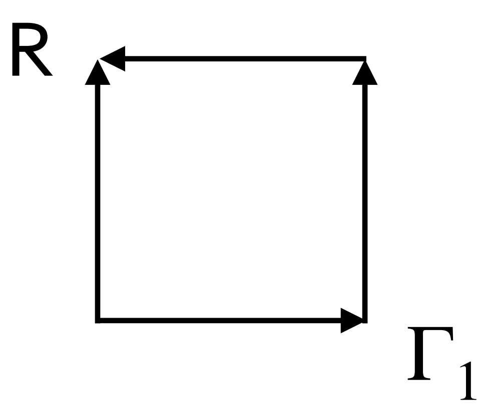

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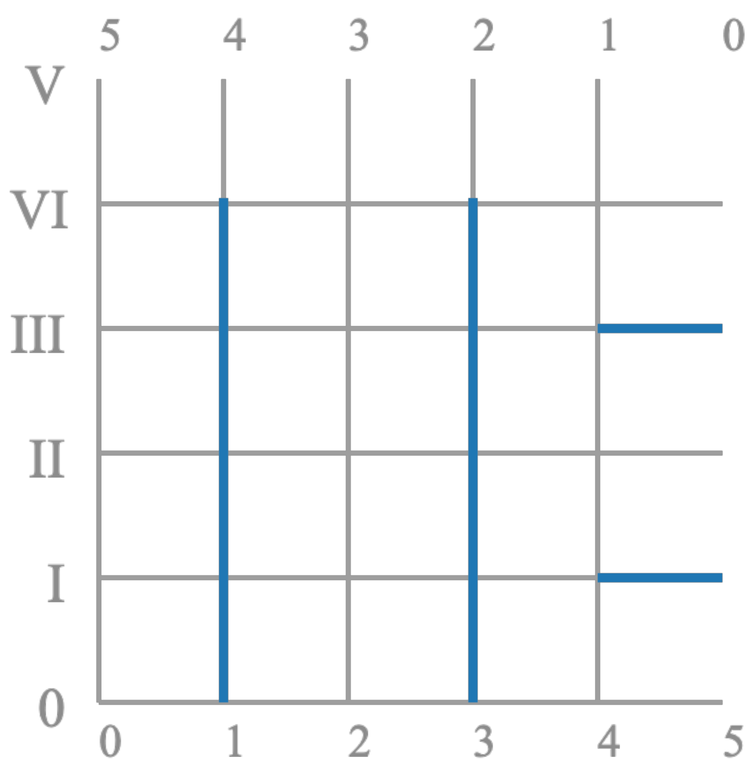

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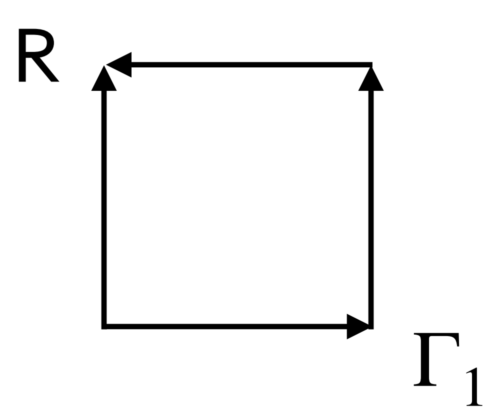

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$$\Xi(-1)^F = (-1)^{\frac{N_f}{2}}(-1)^F\Xi$$

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- It is part of the spacetime π-rotation and itself can't have an anomaly.
- The lattice counter part $T_2\mathscr{R}_2\mathscr{T}$ probes the emergent reflection anomaly on the lattice.

Inequivalent models

Lattice			Cont	Order
Boundary Condition	L_1	L_2	**Twist**	
$(\ell^1, \ell^2) \sim (\ell^1 + L_1, \ell^2) \sim (\ell^1, \ell^2 + L_2)$	even	even	$(1, 1)$	2
	even	odd	$(1, \Gamma)$	4
$(\ell^1, \ell^2) \sim (\ell^1 + L_1, \ell^2) \sim (-\ell^1, \ell^2 + L_2)$	even	odd	$(1, R)$	4
	odd	odd	(Γ, R)	8
	even	even	$(1, \Gamma R)$	4
$(\ell^1, \ell^2) \sim (\ell^1 + L_1, \ell^2) \sim (-\ell^1 + 1, \ell^2 + L_2)$	even	odd		
$(\ell^1, \ell^2) \sim (\ell^1 + L_1, \ell^2 + L_1) \sim (-\ell^2 + L_2, -\ell^1 - L_2)$	even	even		

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Thank You!

Backup Slides

Smith Isomorphism

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anti-unitary $\mathbb{Z}_2$ time-reversal in 0+1d

[Gilkey; Hason, Komargodski, Thorngren; Córdova, Ohmori, Shao, Yan]

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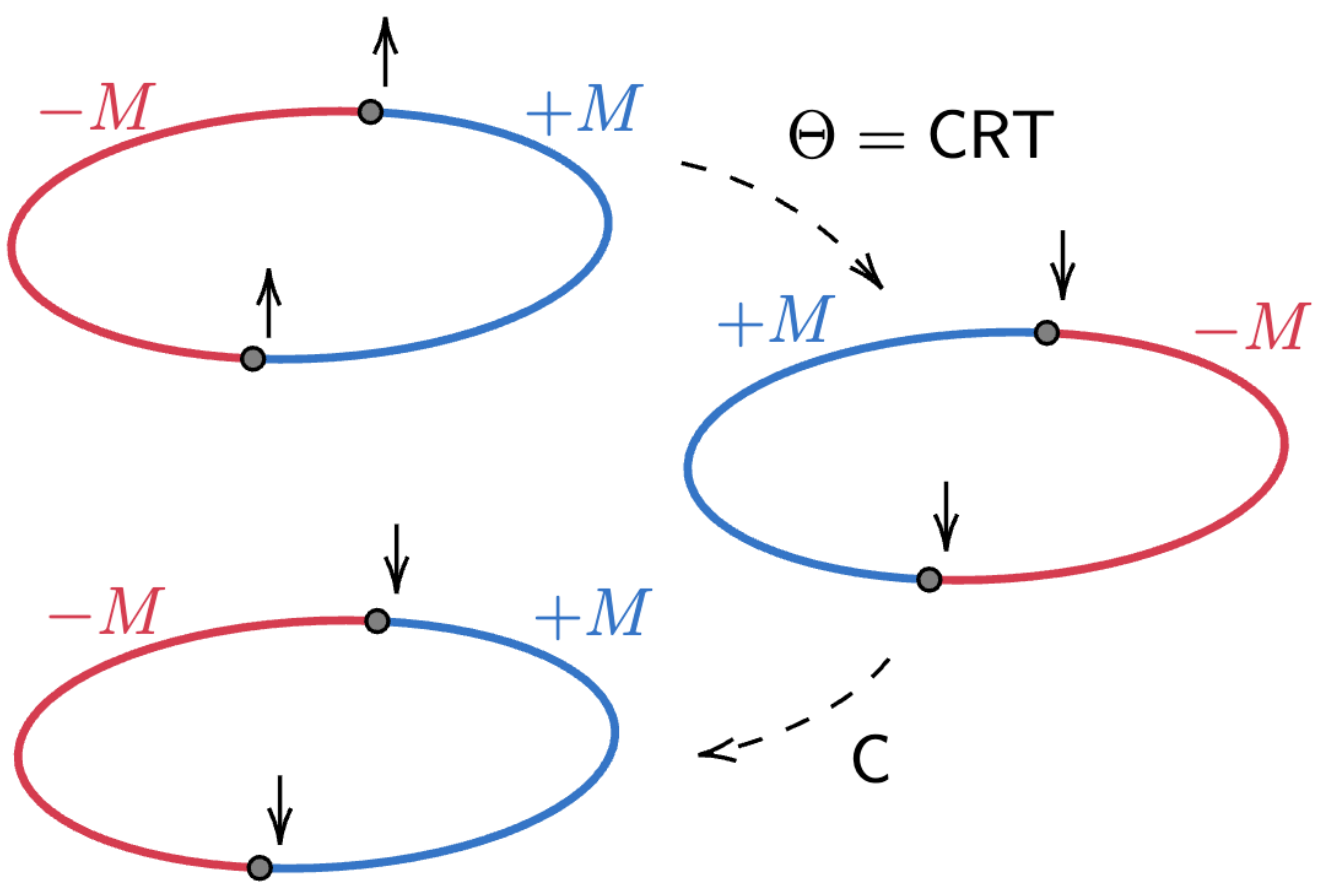

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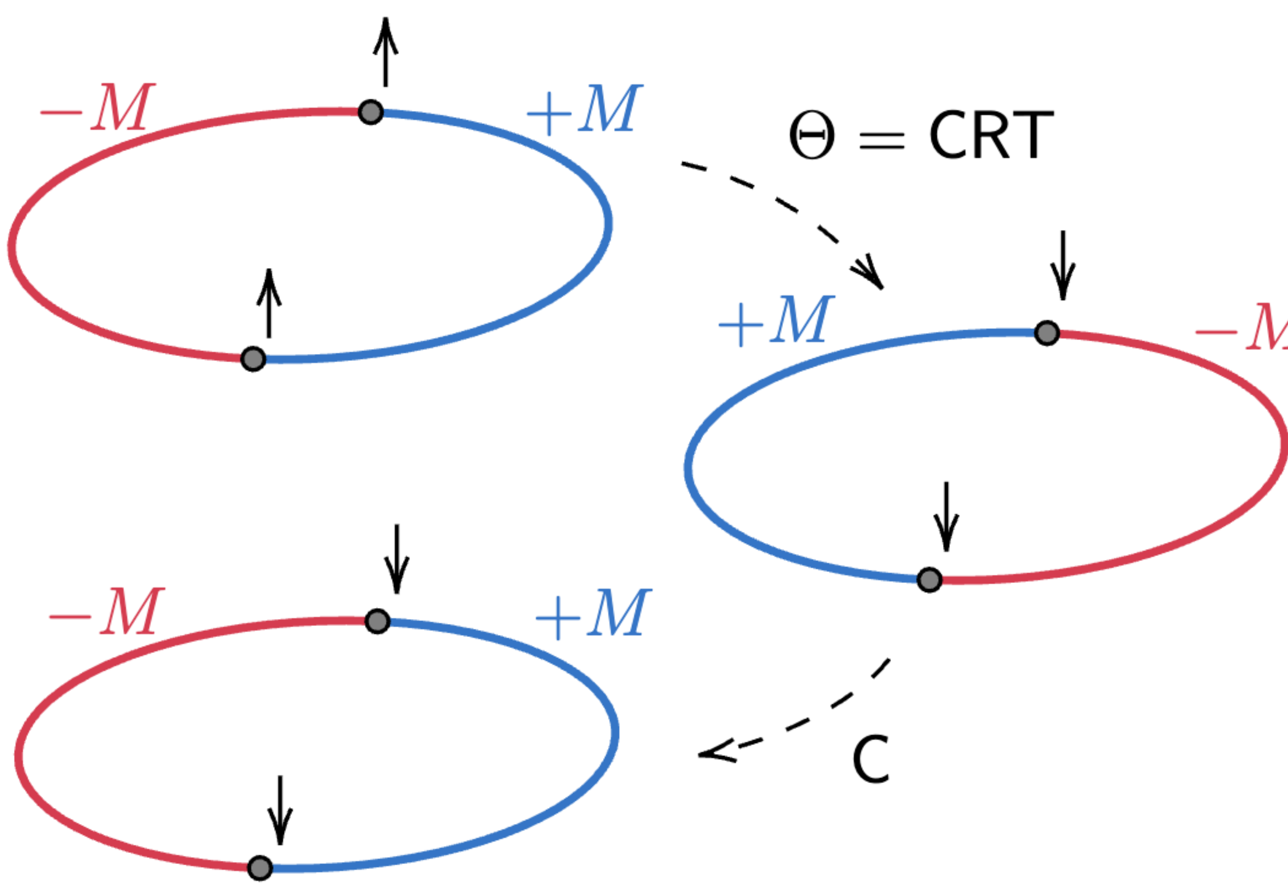

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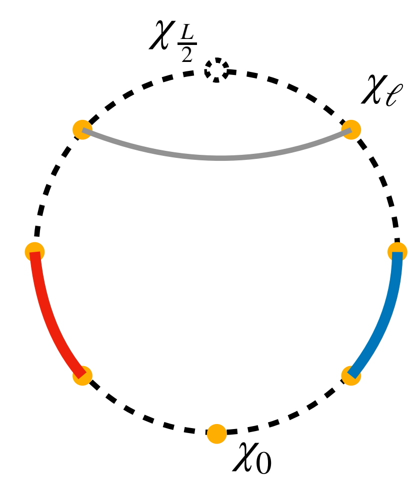

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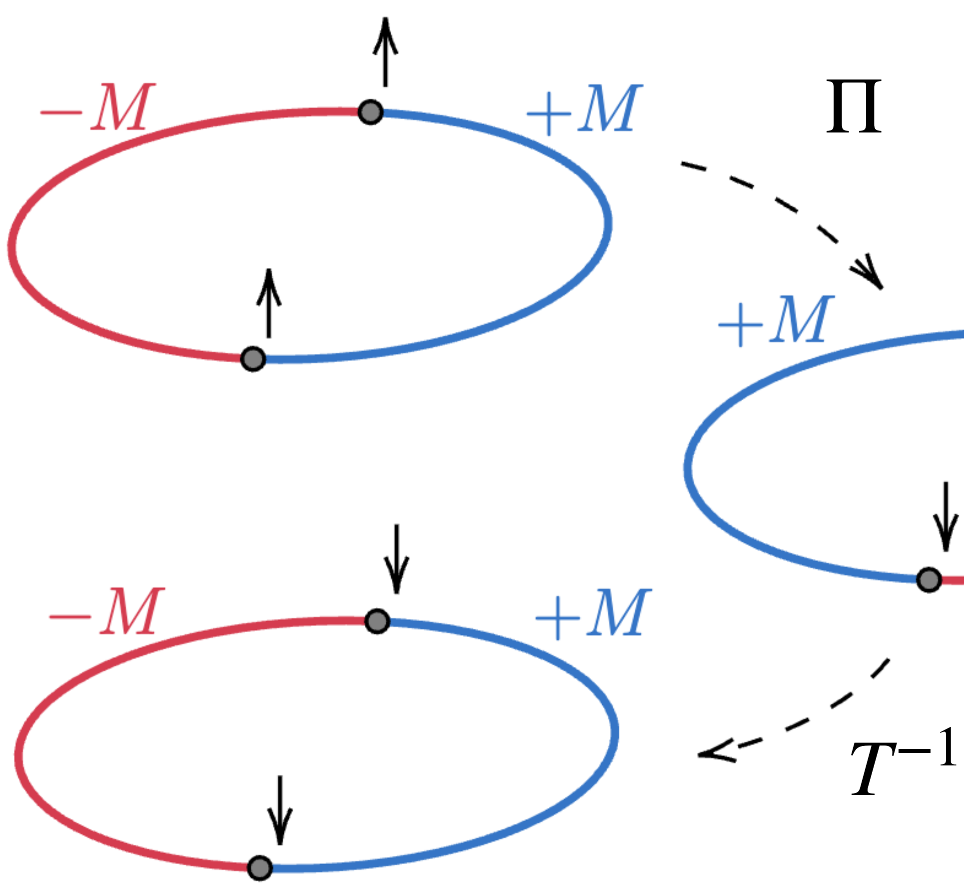

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