Chiral gauge theory and the strong CP problem

Collaborators:
David B. Kaplan (UW),
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, arXiv 2412.02024

Jinlong Dang, Peking University,
Rohith Karur, UC Berkeley
[2606.05306](2606.05306), [2606.05305](2606.05305)

Srimoyee Sen,
Iowa State University

New developments in lattice fermions, 2026

Based on *Phys.Rev.Lett.* 132 (2024) 14, 141604, arXiv 2412.02024

Chiral gauge theories on the lattice

A non-perturbative formulation of such theories is desirable.

Why? It's a question of principle. Can it be done or not?

Can allow us to simulate SM and also BSM models with strong chiral gauge sector.

Phenomenological questions?

May end up teaching us something about how the world works!

Takeaway message before I begin

Many approaches (Luscher's for Abelian CGT, **domain wall based** (Kaplan, Grabowska, Sen, Dang, Shamir), **Symmetric mass generation** (Xu, You, Wen, Tong, Catterall, Chandrsekharan, Hasenfratz), **bosonization** (Cherman, Berkowitz, Jacobson, Chen, Neuzil, Sefnashri, Shao, Singh), **Bok-Golterman-Shamir** approach, **symmetry entangler** (Fidkowski, Throngren))

Significant recent developments in the domain wall approach, eliminated all the known roadblocks to a CGT formulation.

QCD formulated this way does not suffer from the strong CP problem!!

That's the whole talk!

Clarification

- We have lacked a formulation itself. -- This is where the longstanding bottleneck has been.

- But there is another problem. Even after a formulation is obtained, practical Euclidean computation will require Monte Carlo sampling:

  there is also a sign problem!

  -- we will eventually need a quantum simulation if we want to extract observables.

Plan of the talk

- What are chiral gauge theories?
- Why is it hard to formulate them on the lattice?
- What is the domain wall based approach? A few of the past attempts, that are yet to work or don’t work.
- What is new and why it could be the best domain wall based bet.
- Why does QCD formulated this way does not suffer from strong CP.
- Path forward?

Chiral gauge theories

Even dimensional world with massless fermions and gauge field.

Chiral symmetry is gauged.

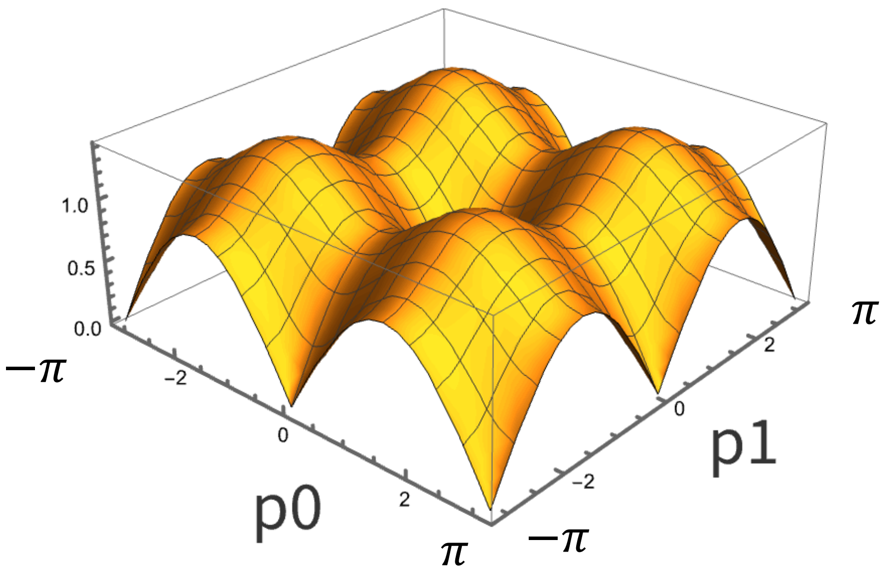
Fermion mass terms transform under gauge transformation.

So, a simple mass term is disallowed.

Example: The standard model.

Good chiral symmetry is essential

Backing up: Even global chiral symmetry is hard (but solved)

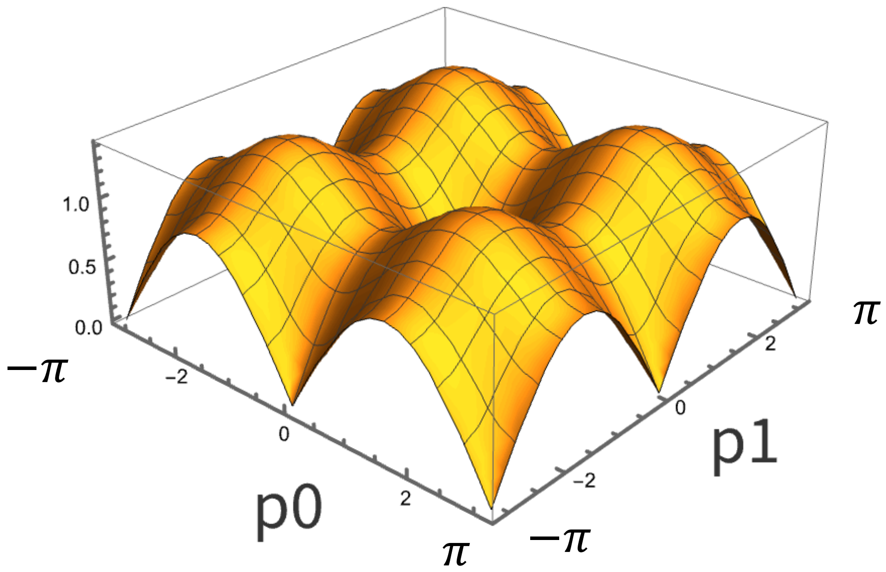


Lattice Brillouin zones

Simulating massless Dirac fermions on the lattice, was a challenge due to fermion doubling.

Nielsen Ninomiya (1981): Cannot formulate Dirac fermion with exact chiral symmetry without an unwanted doubling of all fermion species.

What does this mean ? (still talking about global chiral symmetry)



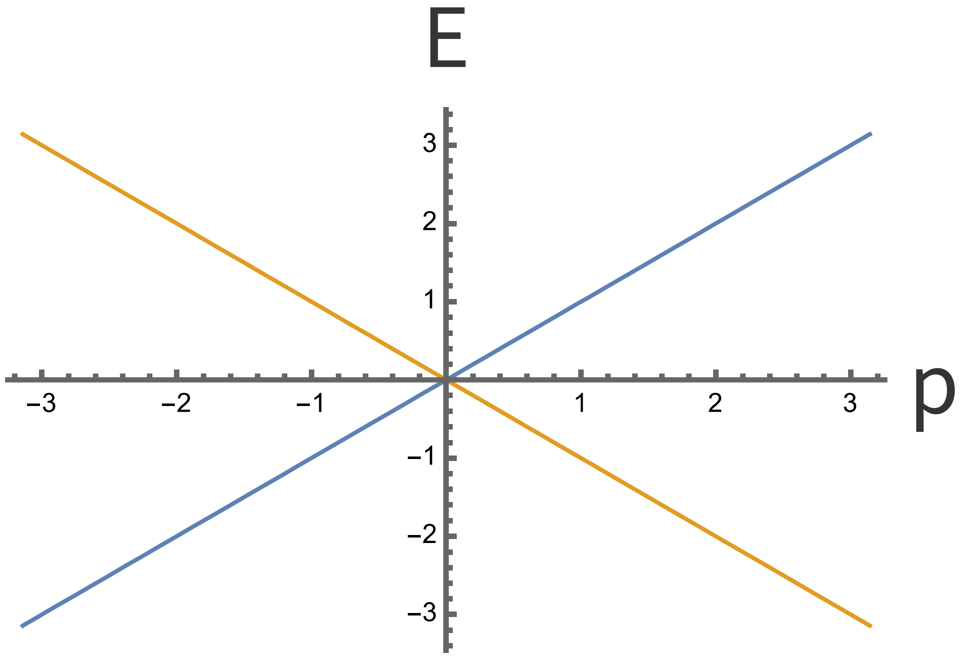
Lattice Brillouin zones

We want to formulate a Dirac fermion theory on the lattice.

We want to keep the fermions light, e.g. light quarks.

This combination i.e. “discrete space-time + massless fermion”: extremely difficult to engineer.

Why chiral symmetry is hard: Dispersion (1 spatial dimension)



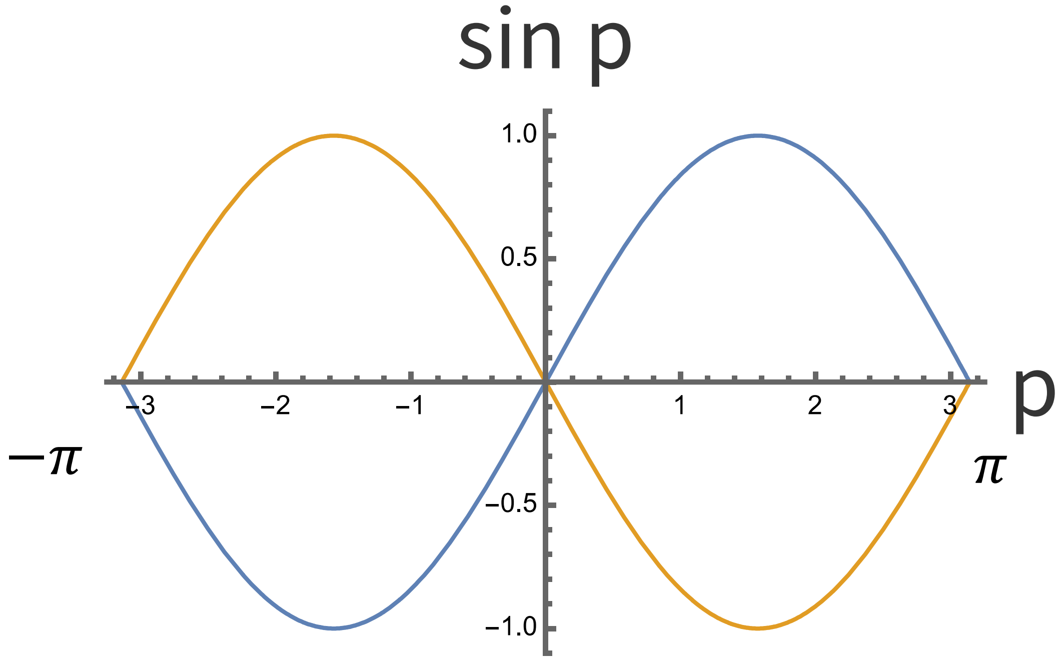
Continuum dispersion for a Dirac Hamiltonian

The no-go is better visualized using dispersion relation in Minkowski space-time (time continuous).

Hamiltonian formulation.

$$E = \pm p$$

Brilliuoin zones (Dirac)



Two Dirac fermions

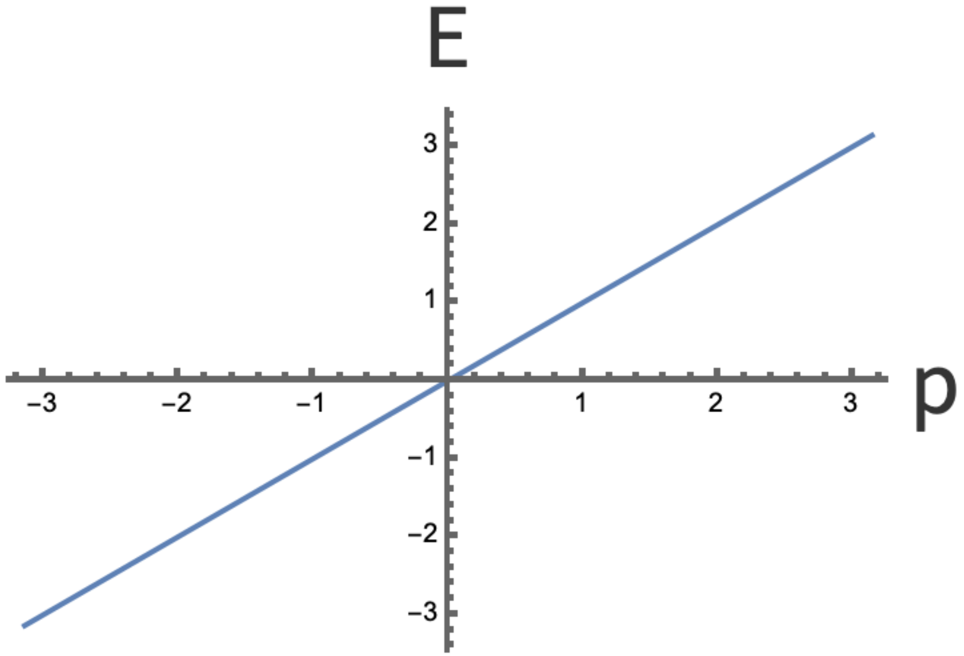
Lattice in space.

Time not discretized.

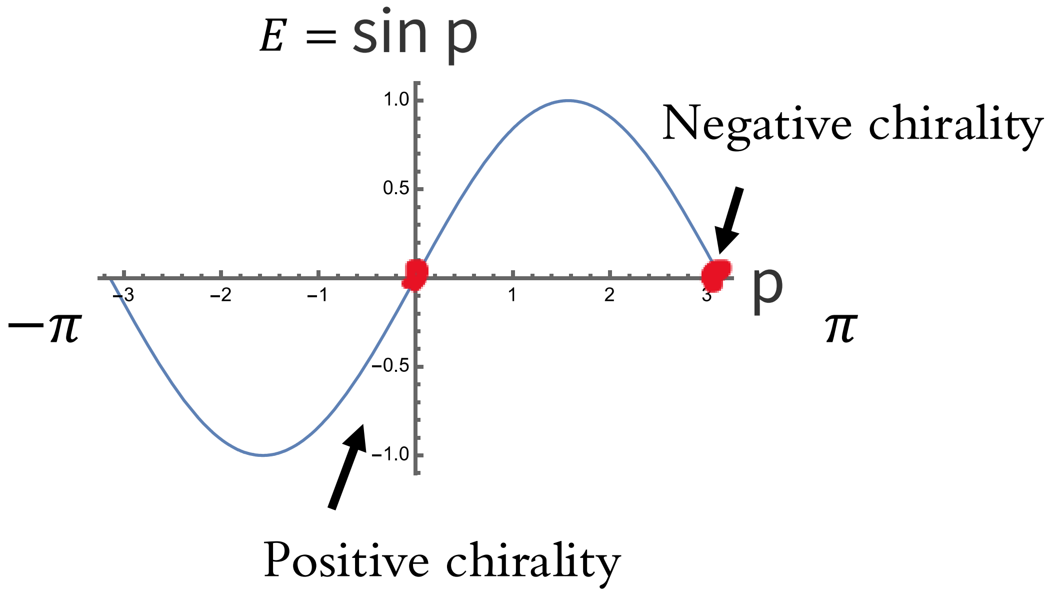
Solving the naively discretized Dirac Hamiltonian with eigenvalues $\pm\sin p$

$$E = \pm \sin p$$

Weyl fermion



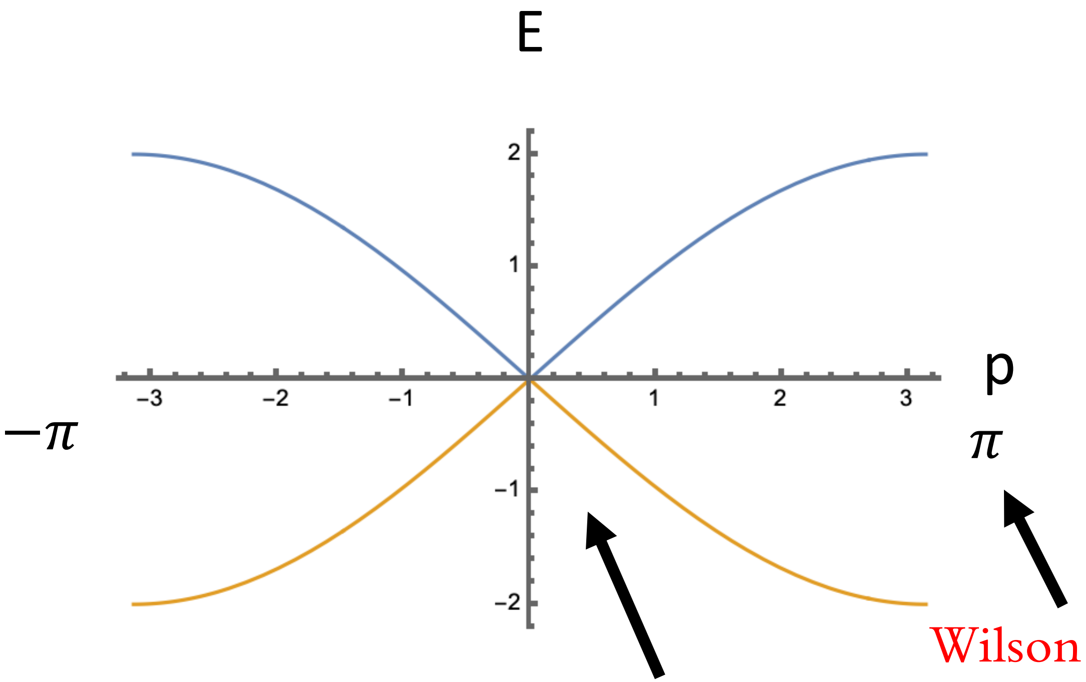
Continuum



Lattice

Even number of zero crossing of periodic functions

Wilson term for Dirac



Lattice in space.

Time not discretized.

Solving the naively discretized Dirac Hamiltonian with eigenvalues $\pm\sin p$

Wilson term removes this.
But kills chiral symmetry

$E = \pm \sin p$

Gaplessness is not protected

∇_1= Symmetric finite difference in space
∇= symmetric discrete spatial Laplacian

Single particle Hamiltonian: $\mathrm{H} = -i\gamma^1\nabla_1 + \boxed{m + \frac{R}{2}\nabla}$

$\nabla \rightarrow (1 - \cos p)$

Loss of exact chiral symmetry

Quantum corrections kill massless-ness when interactions are turned on.

Undesirable if we want massless fermions.

Domain wall fermions

Domain wall fermions can solve this:
by separating the two chiralities in space,
just like the edges of a quantum Hall state.
(Kaplan, 1992)

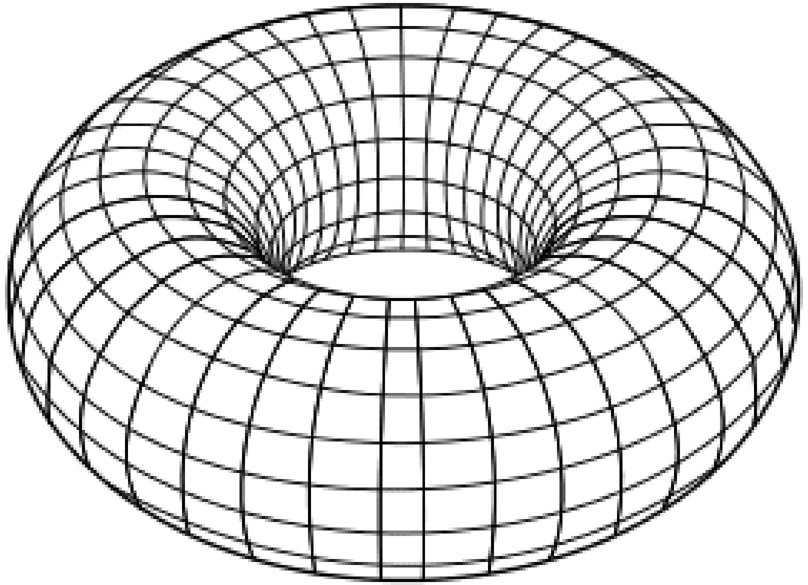
Kaplan, Shamir, Narayanan (overlap), Neuberger (overlap), Luscher

How do you do this?

Go to higher dimension. Consider 2+1 D Dirac fermion. Let's go to finite volume.

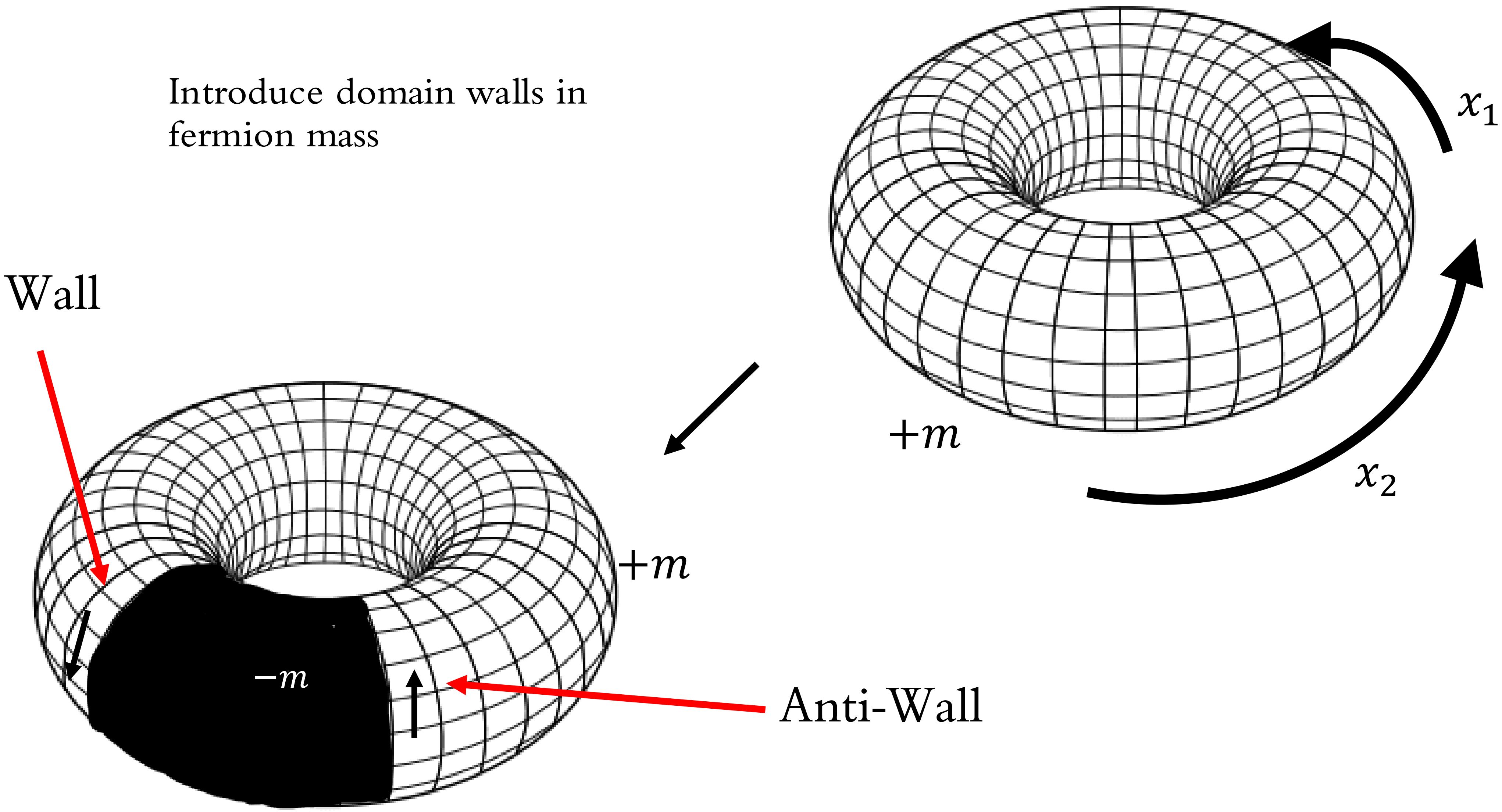
Periodic boundary condition in all directions with uniform Dirac mass.

$+m$

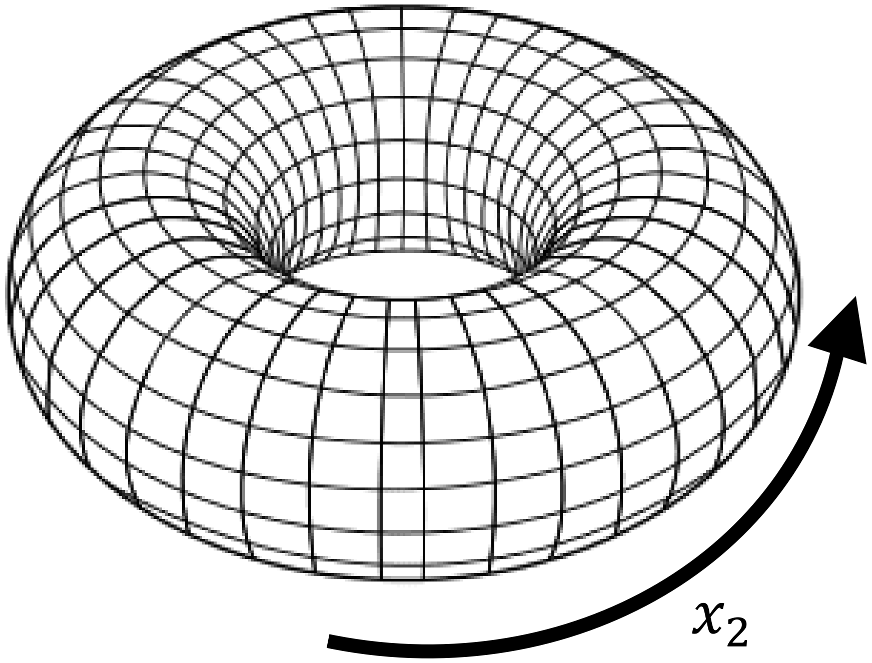


How do you do this?

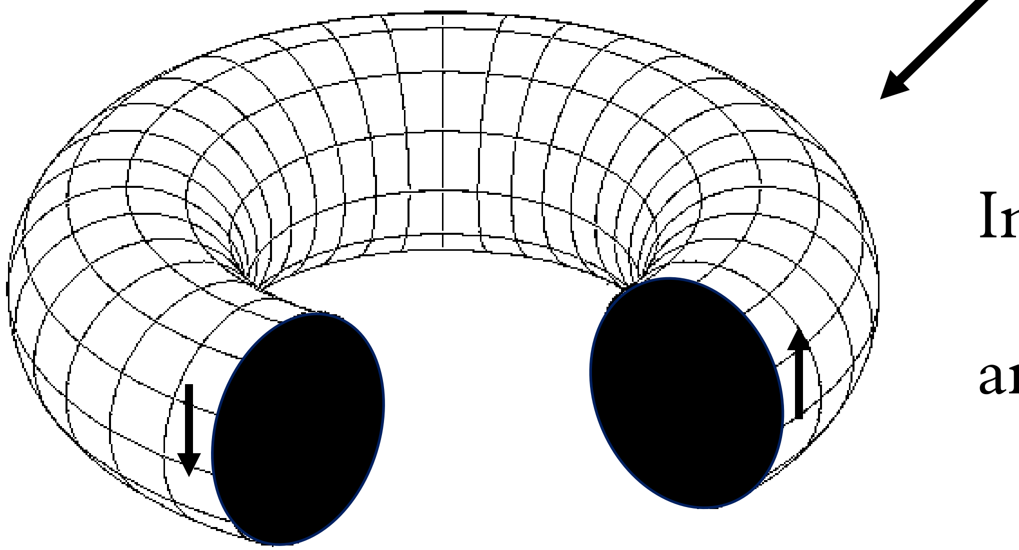
Introduce domain walls in fermion mass



How do you do this?



Standard domain wall
fermion defect



In fact, we can get rid of the negative mass region

and work with open boundary condition in x_2

Shamir, *Nucl.Phys.B* 406 (1993) 90-106

γ_2 becomes the chirality matrix

On the lattice

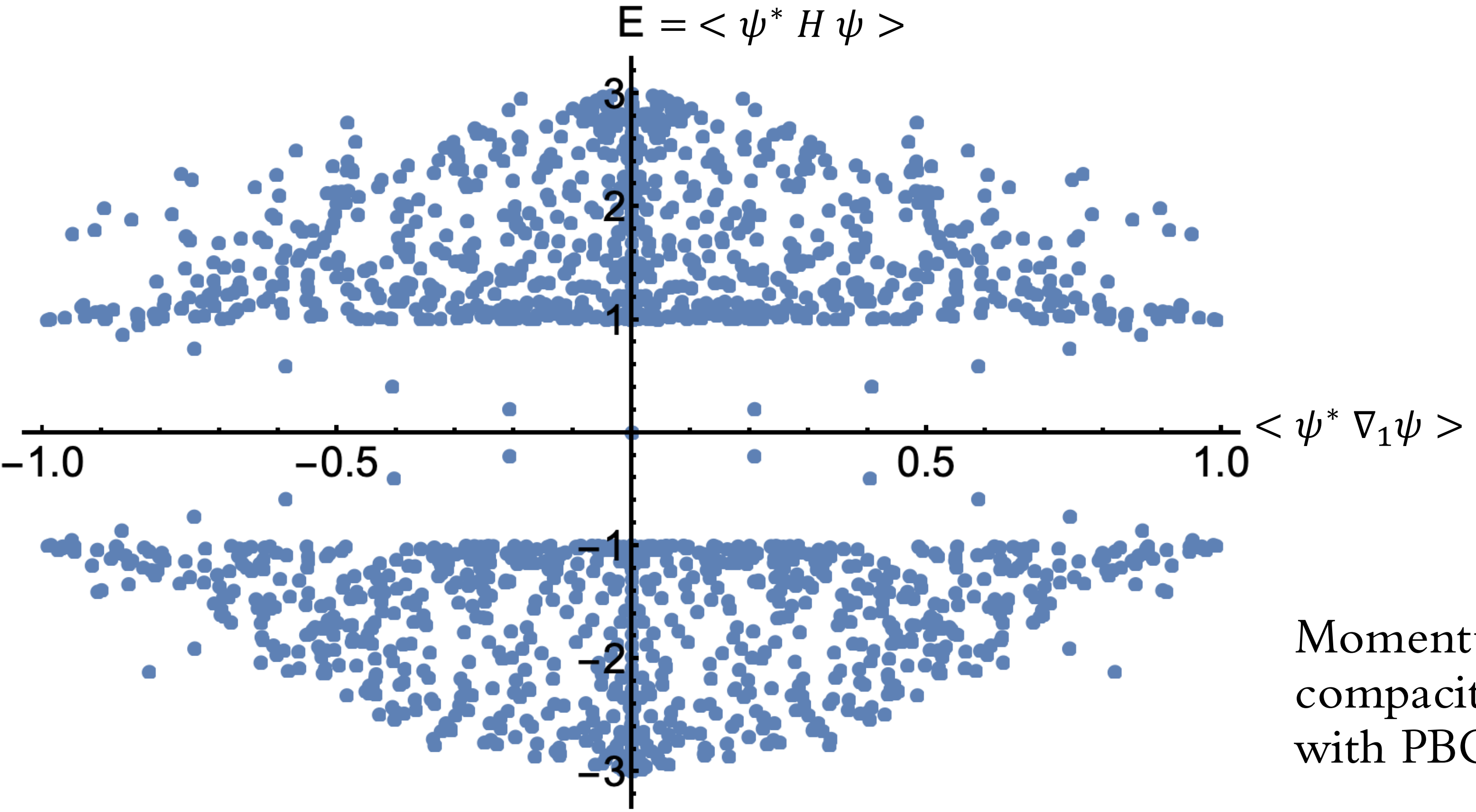
It's the Wilson fermion with no discretization in time.

Single particle Hamiltonian: $\mathrm{H} = -i\gamma^i\nabla_i + m + \frac{R}{2}\nabla$ with $i = 1,2$

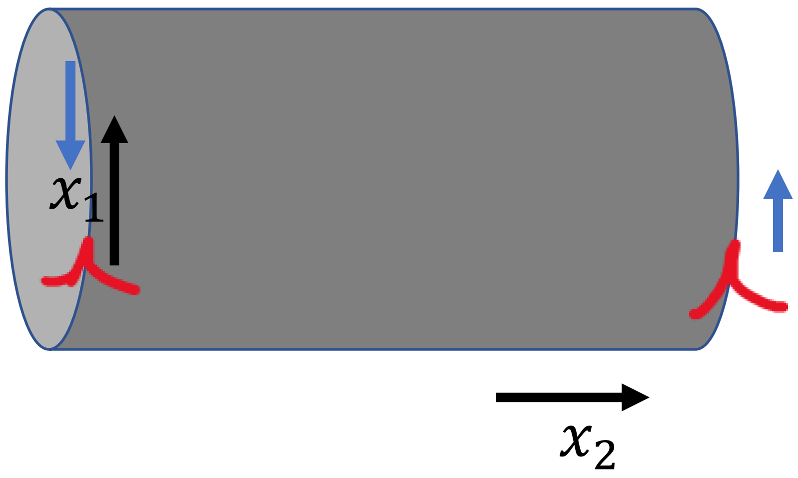
∇_i= Symmetric finite difference in space
∇= symmetric discrete spatial Laplacian

Look at the spectrum

Domain wall fermions on the lattice



Opposite chiralities



Momentum along the compacitified dimension with PBC

Solved using domain wall fermions

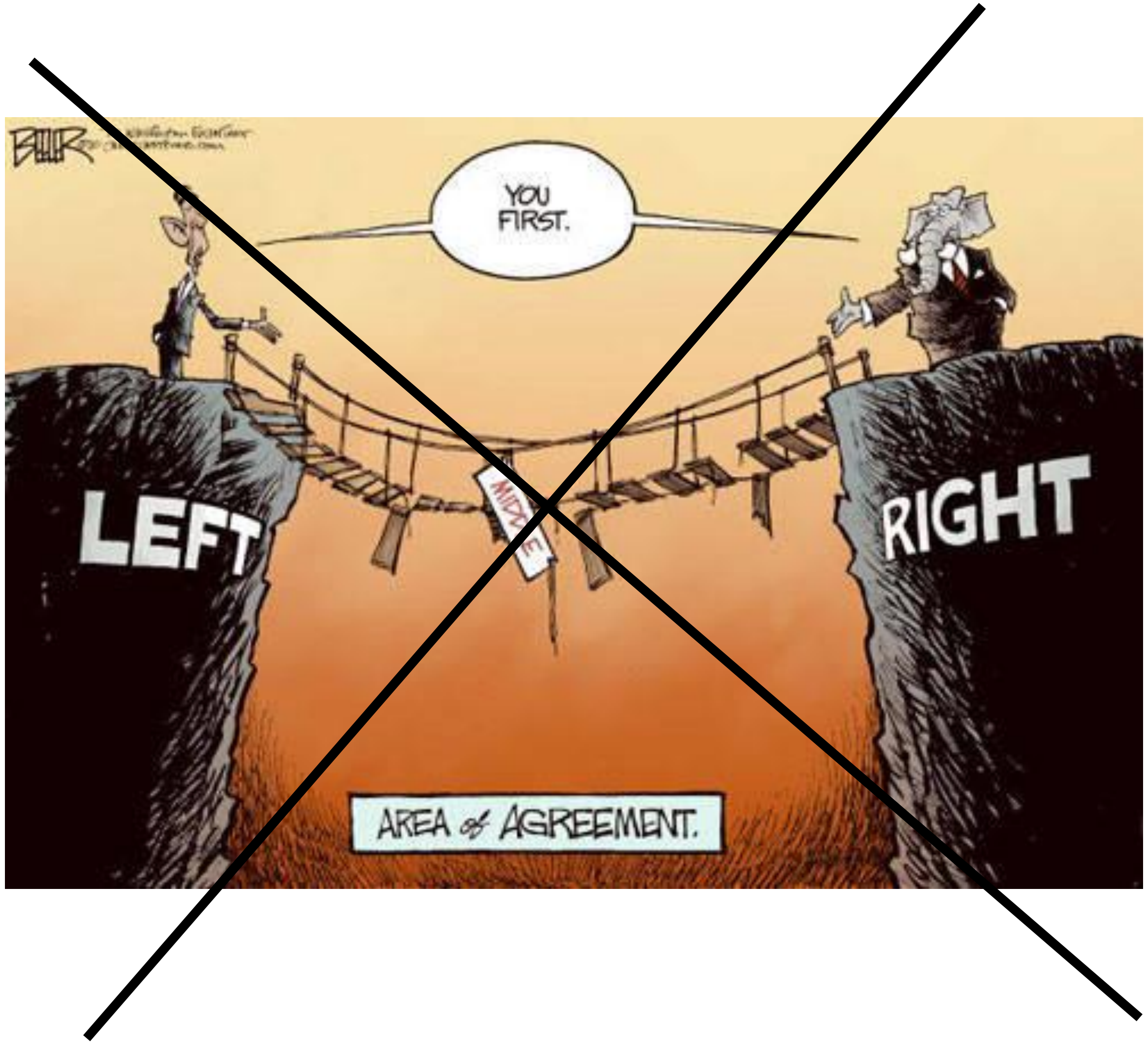
- Right and left moving modes separated in space. So, any quantum correction to mass exponentially suppressed.
- Allow gauge fields to talk to both walls in the same way producing a vector gauge theory.
- Very useful in QCD simulations.

Doesn’t work for chiral gauge theories

The idea does not work for chiral gauge theories though.

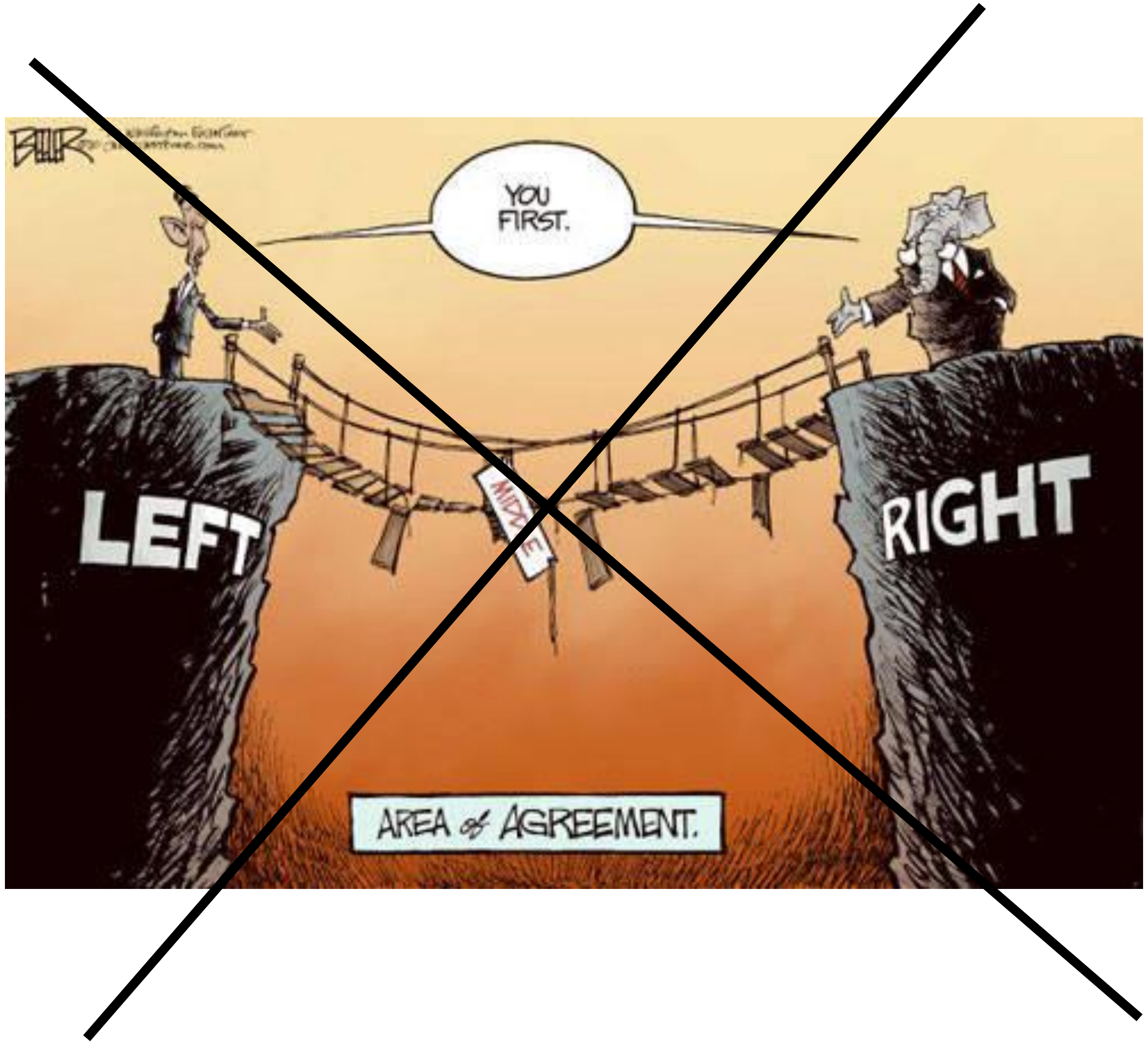
The construction in finite volume necessarily has two defects.

Two defects lead to opposite chiralities producing vector theory.



Doesn't work for chiral gauge theories

We need to isolate Weyl fermions of a particular chirality --- impossible with the standard domain wall setup.

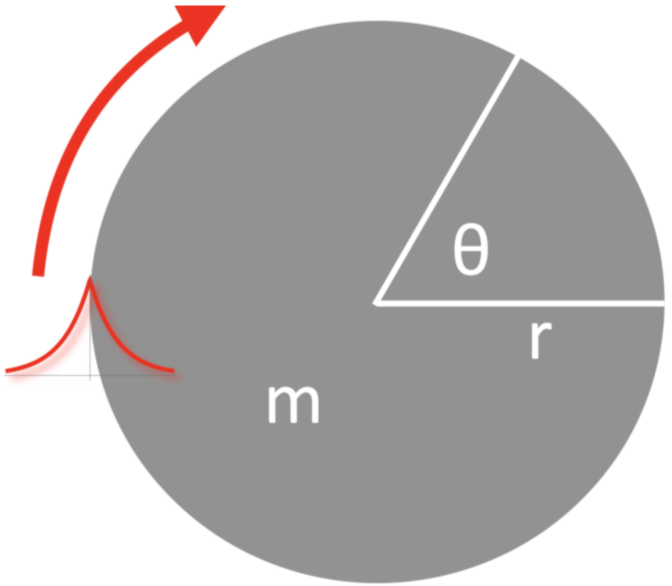


A different approach

Why have two defects?

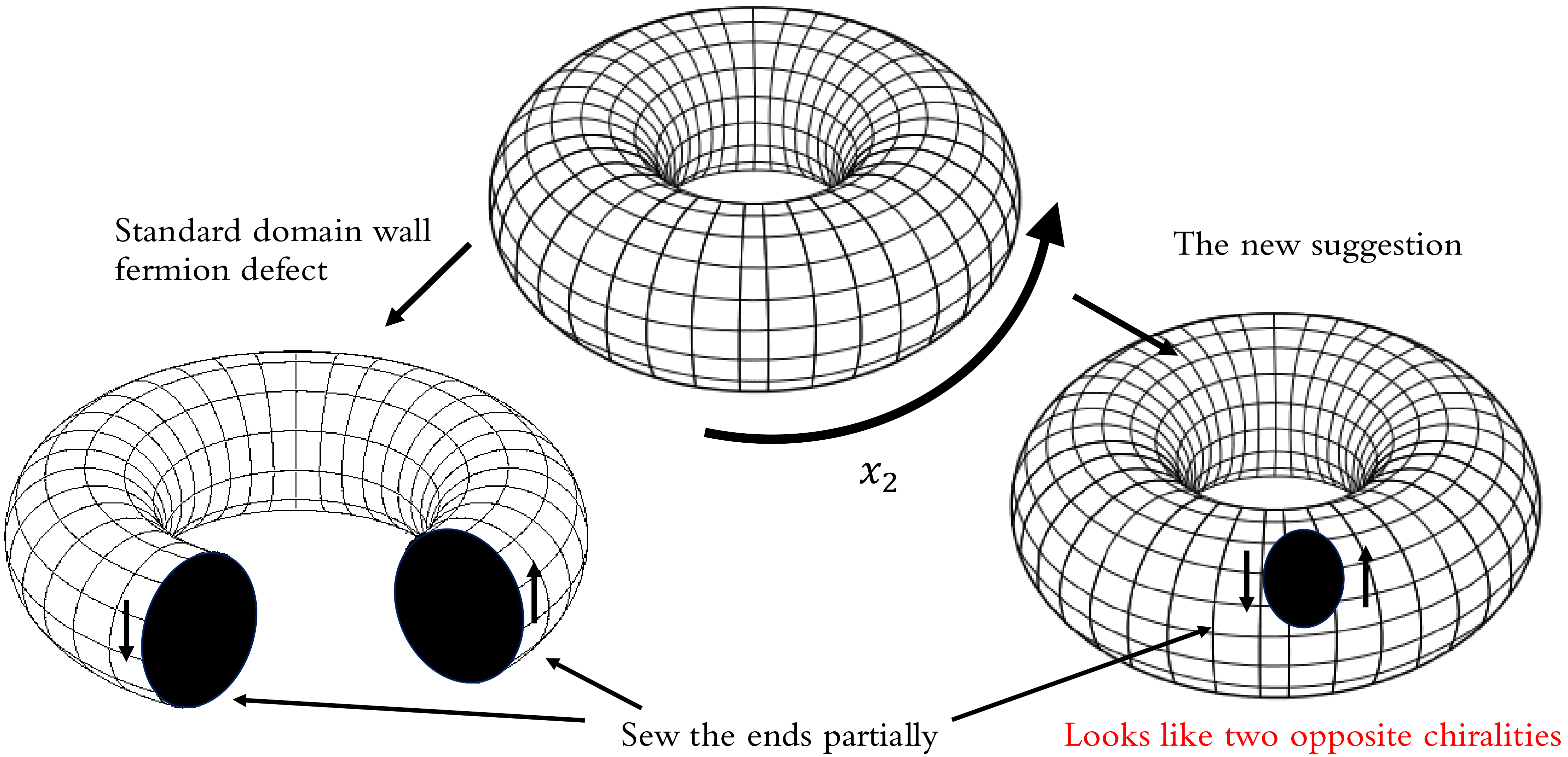
Have one.

Use a manifold with a single defect. E.g. a disc.



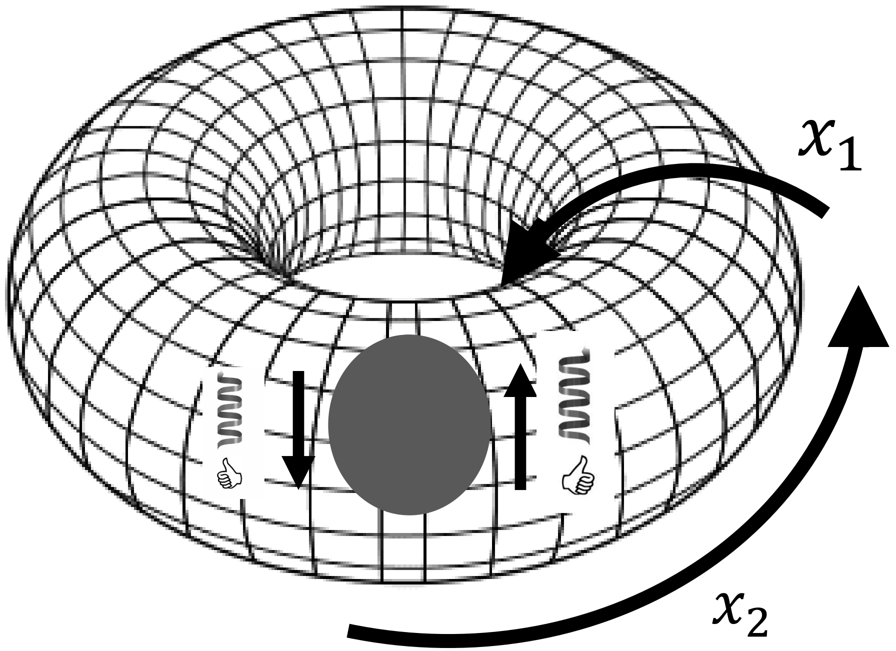
There were reasons to think this might not work

What does the disk mean?



Opposite chirality on the two sides..

Maybe we are not thinking about chirality the right way.

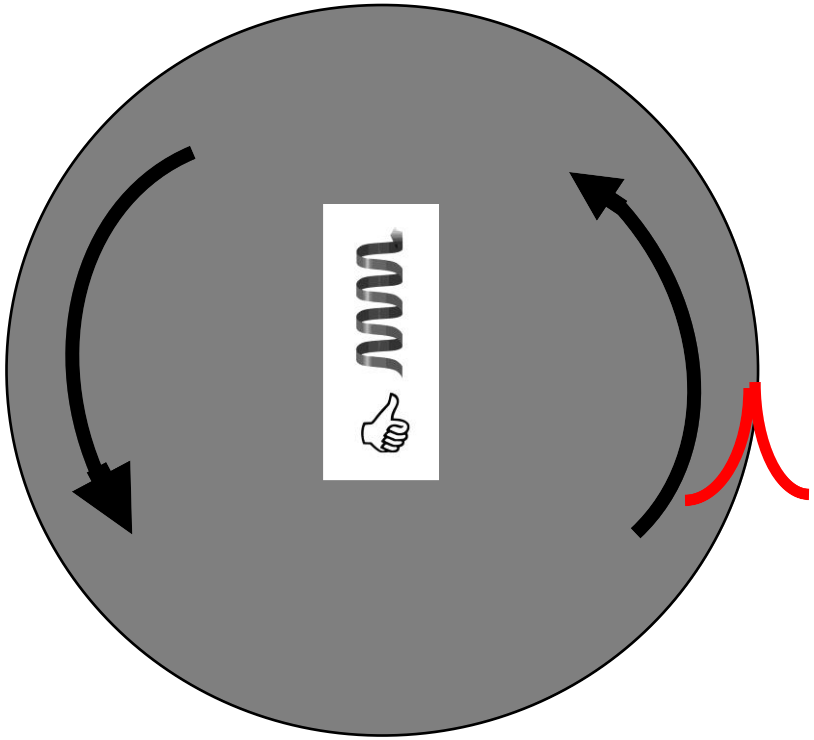


Define chirality in a position dependent manner

Define chirality as clockwise travel vs anticlockwise travel:

counter-clockwise

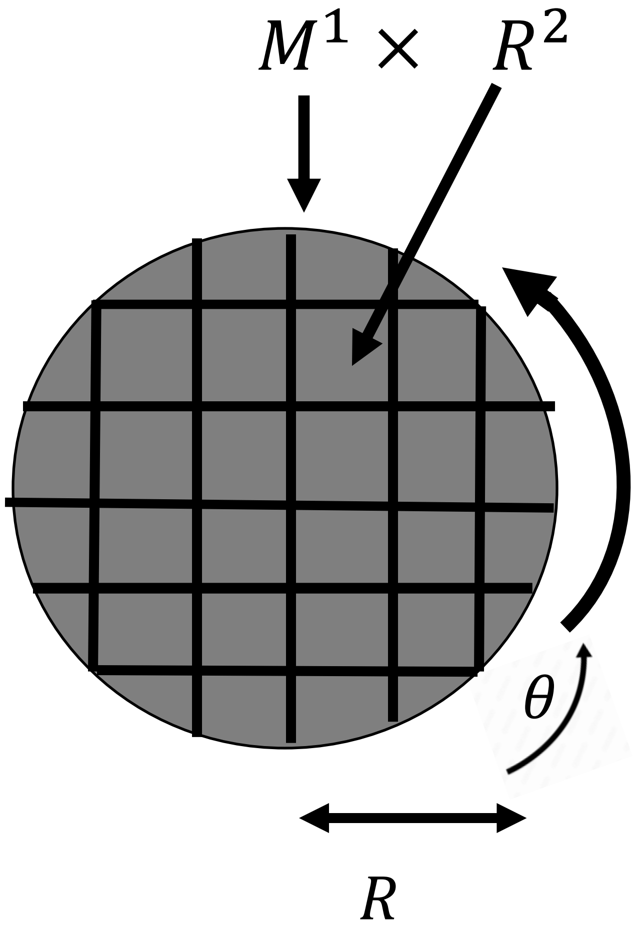
clockwise



Single chirality: Weyl mode

Phys.Rev.Lett. 132 (2024) 14, 141603 (Kaplan)

What do you see on the lattice?



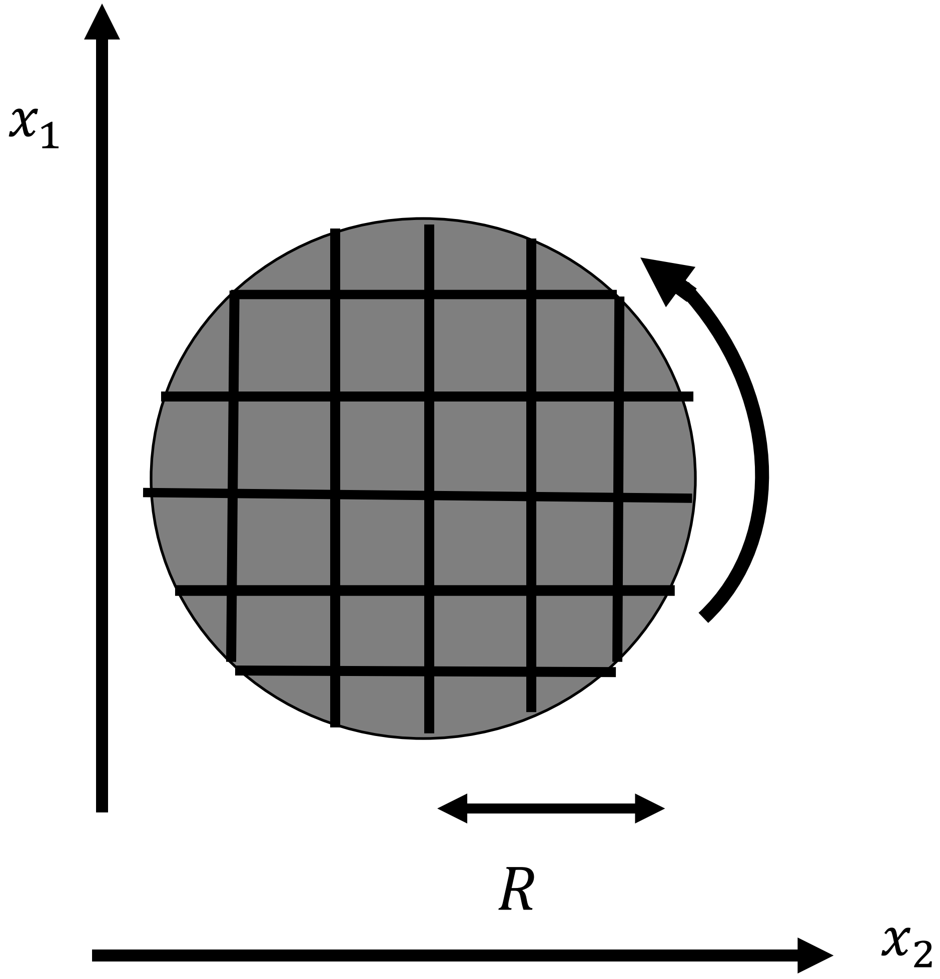
Take M^1 to be **time** and don't discretize it. Discretize r, θ (or x_1, x_2)

Find the eigenvalues of the discrete space Hamiltonian of the full theory.

What do we expect to see?

Kaplan, Sen, *Phys.Rev.Lett.* 132 (2024) 14, 141604

Disc



Check the dispersion.

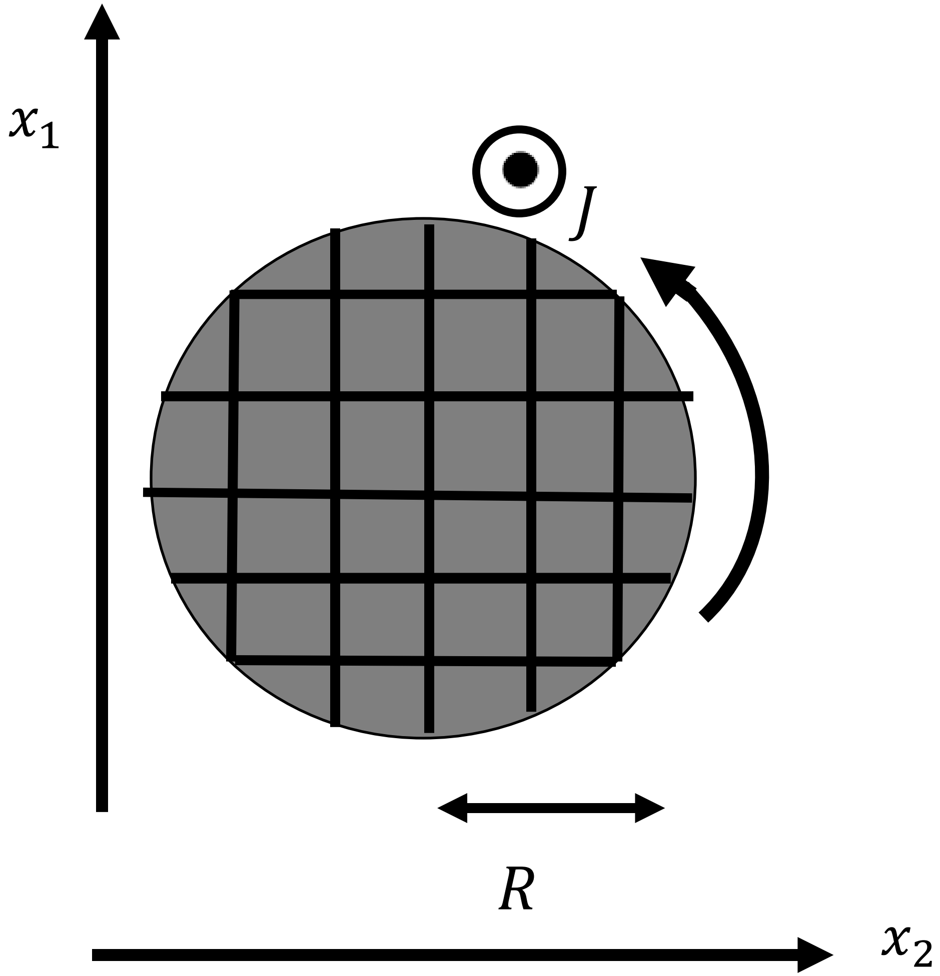
How?

Broken translation invariance along both x_1 and x_2

Does not make sense to plot E vs p_1

Kaplan, Sen, *Phys.Rev.Lett.* 132 (2024) 14, 141604

Disc



We have rotational invariance (approx).

Diagonalize the lattice Hamiltonian.

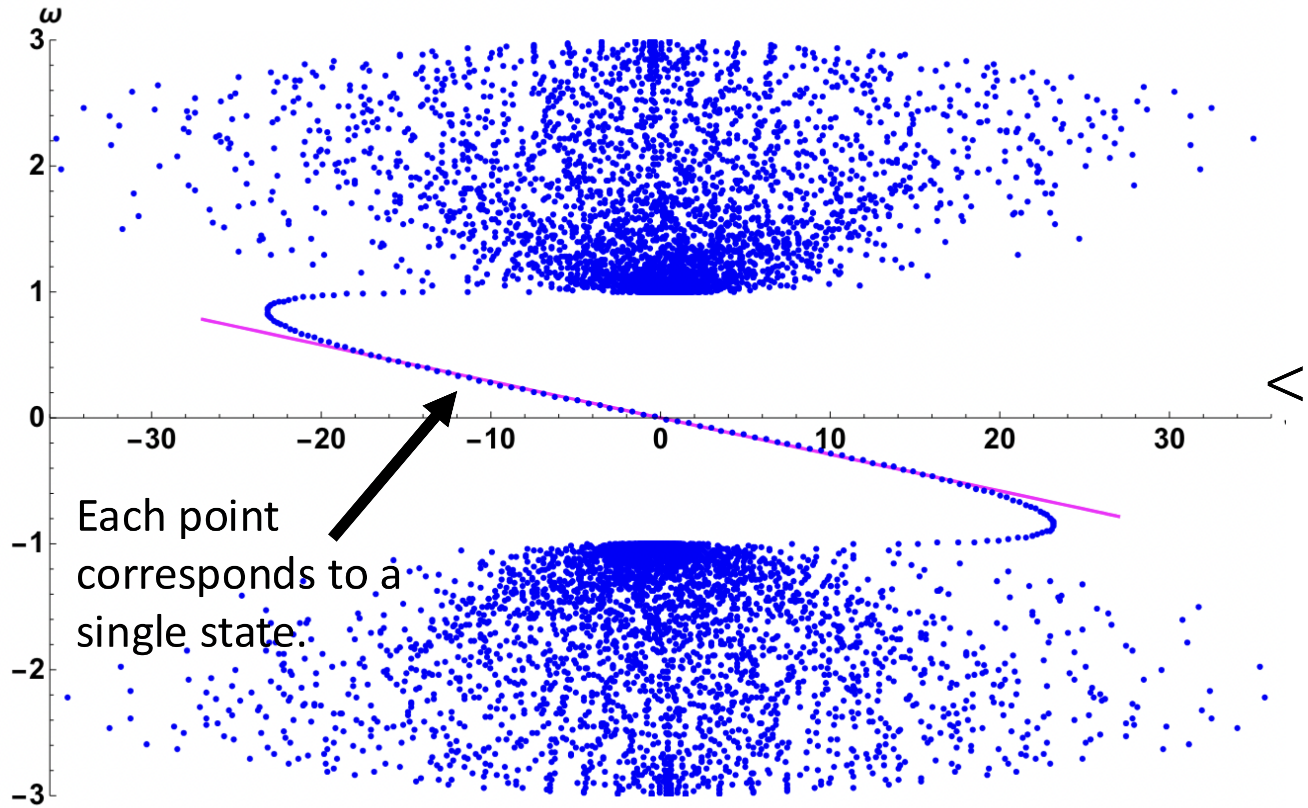
Compute expectation values of angular momentum J

Plot E vs J

Kaplan, Sen, *Phys.Rev.Lett.* 132 (2024) 14, 141604

Dispersion for the disk

$E = <\psi^* H \psi>$



$<\psi^* J \psi>$

Exactly as expected from the continuum

Disk of radius $R = 34$ in lattice units.

Linear dispersion:

$$E = -J/R$$

Kaplan, Sen, *Phys.Rev.Lett.* 132 (2024) 14, 141604

Two tasks for CGT

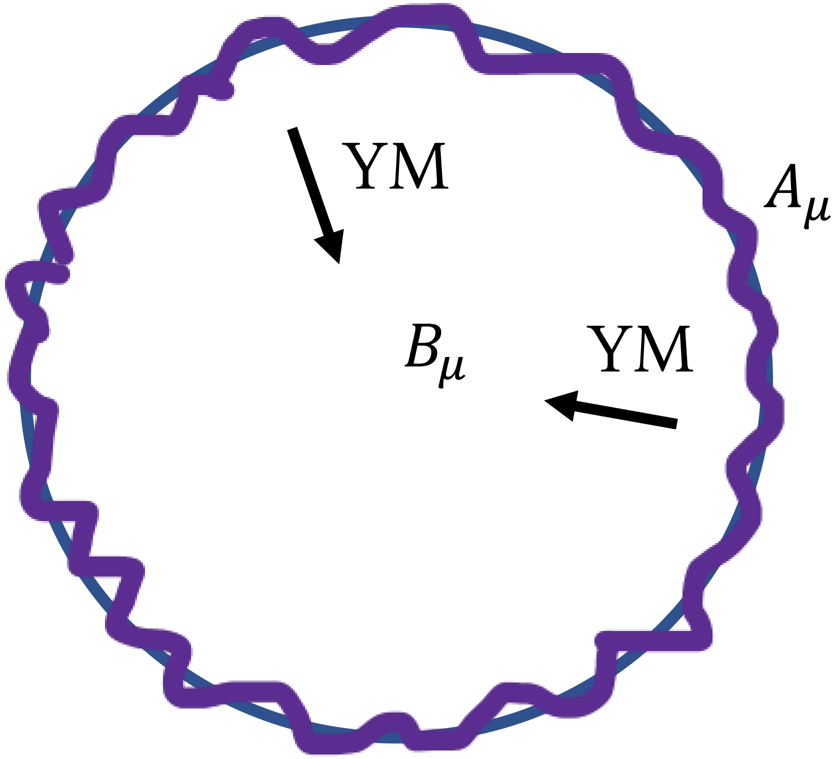
- Isolate Weyl fermion content
- Proposal for gauging Next.

This step is done.

Gauging

Want a $d = 2, 4$ dimensional gauge field and not $d + 1 = 3, 5$ dimensional one.

$d = 2, 4$ dimensional gauge field B_μ in $d + 1$ dimensional bulk.



Integrate over the boundary gauge field A_μ

Bulk gauge field satisfies equations of motion (e.g. YM) while matching A_μ on the boundary.

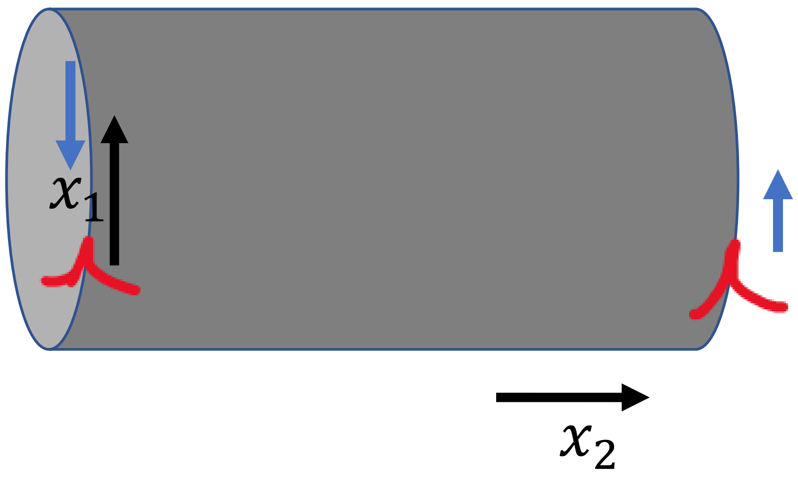
Gauging

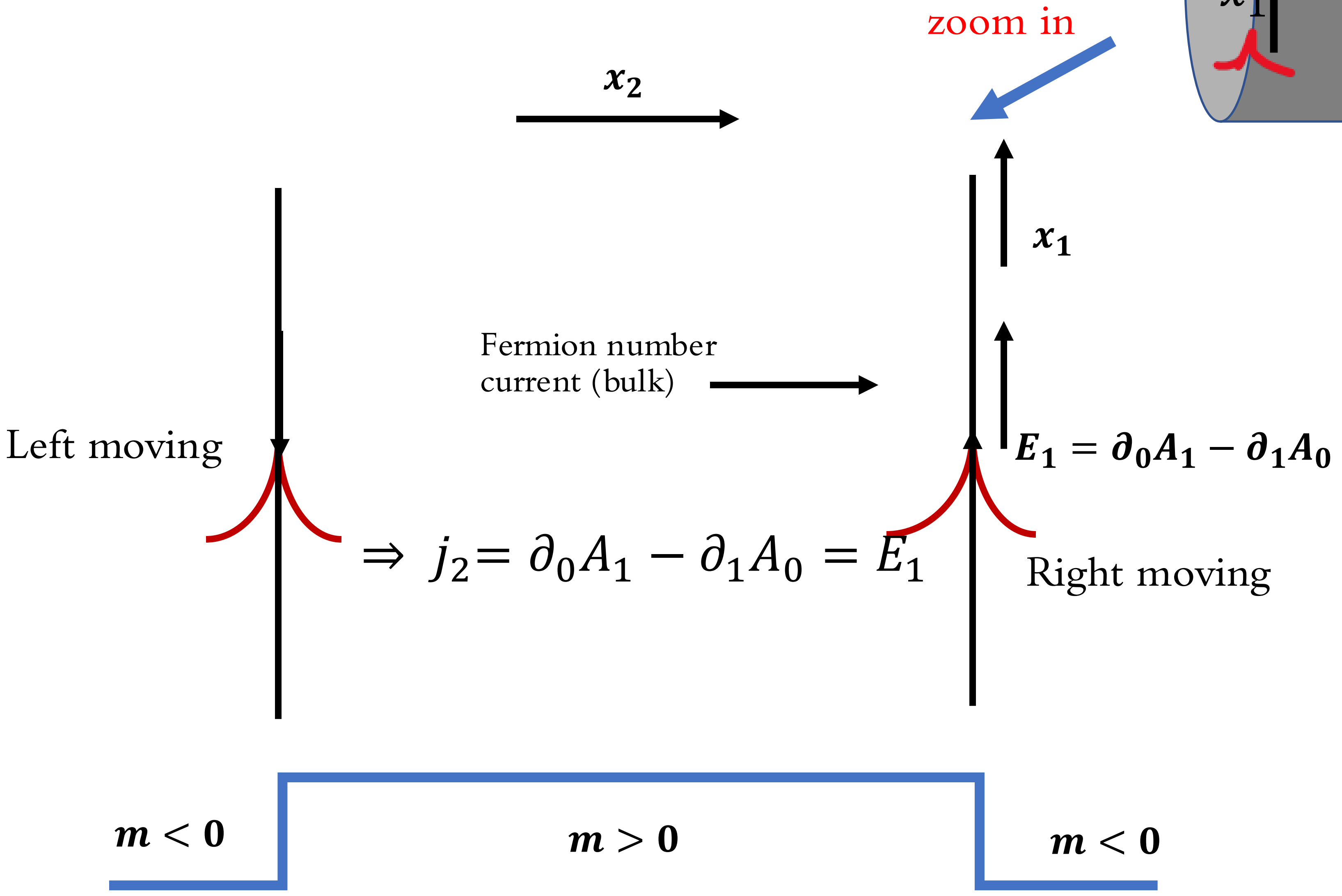
Can engineer any number of Weyl fermions on the boundary.

We can gauge any subgroup of the available global symmetry of the free fermion theory. ---- Must make sense only if the theory is anomaly free.

How do we see this?

Domain wall + anomaly



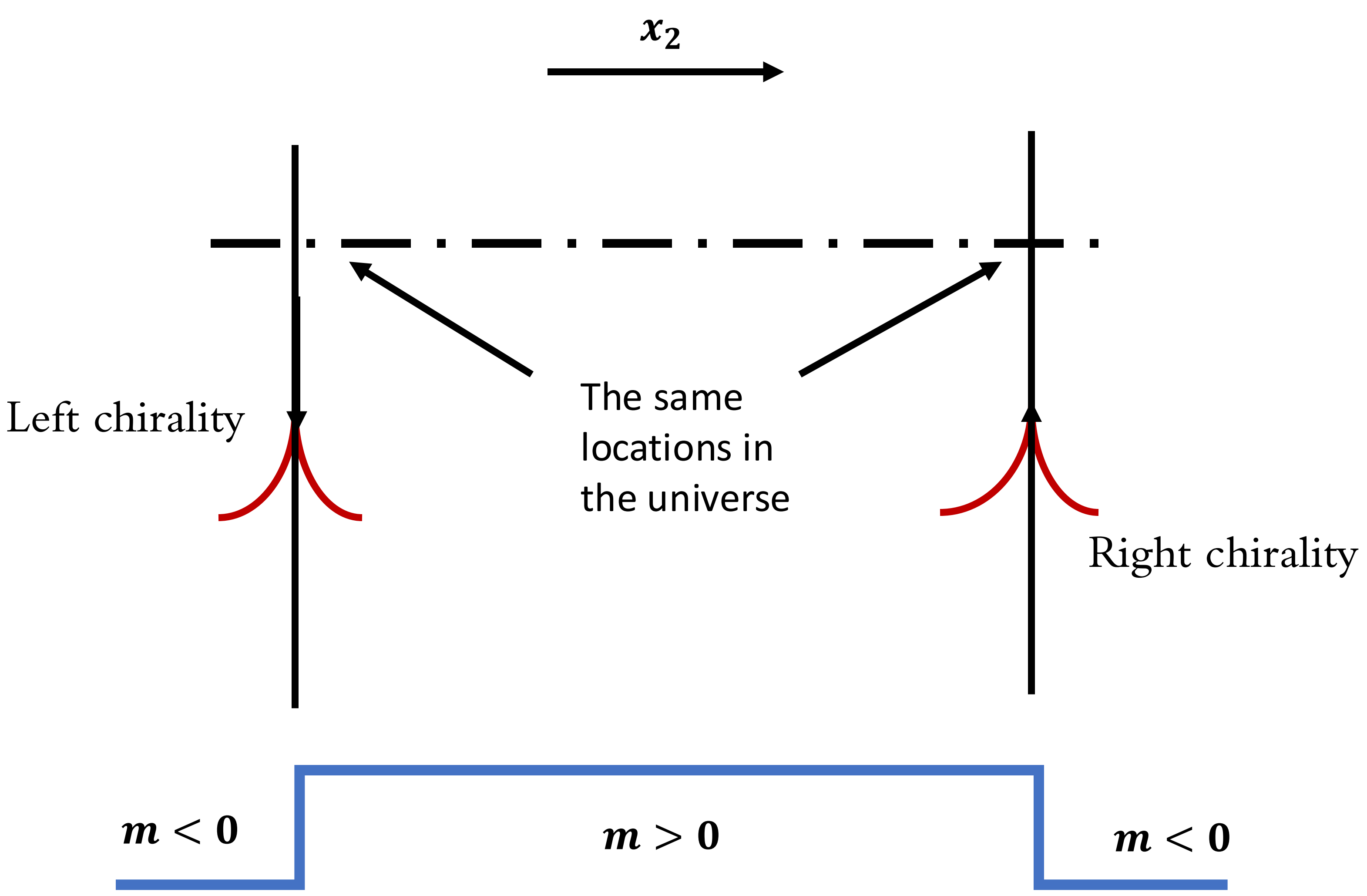


2 + 1 D Dirac fermion setup with domain walls

Gauge the Dirac fermion

Bulk talks to the boundary, great for QCD

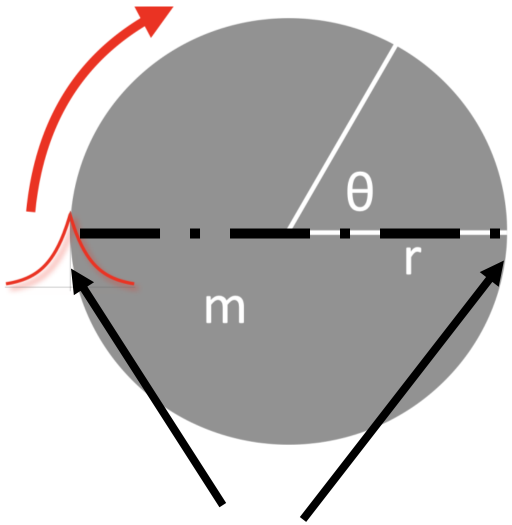
Domain wall + anomaly + QCD



Bulk talks to the boundary, great for QCD

The two walls talk to each other

disk + anomaly



Different locations in the universe, shouldn't communicate across the defect

But they will if the boundary theory has gauge anomaly: nonlocal!!

Thankfully, the standard model is anomaly free.

So, the disk construction makes sense, and the boundary theory is local.

More is true as we will see next.

Summary

We have a sensible microscopic theory of a Dirac fermion which at low energy produces a single Weyl fermion on the lattice.

Nielsen Ninomiya is not an obstacle. We were fixated on the wrong kind of defect.

Removes one of the most significant obstacles of realizing a chiral gauge theory in the domain wall approach.

Golterman-Shamir, Fukaya-Aoki observations

Golterman-Shamir: Phys.Rev.D 109 (2024) 11, 114519, Fukaya-Aoki: 2212.11583

Anomalous (ABJ) anomalies/symmetries of the target theory are exact symmetries of the higher dimensional theory.

Consider one–flavor QCD on the boundary, there appears to be two exactly conserved $U(1)$ currents, vector and axial.

For every Weyl fermion on the boundary, we need a Dirac fermion on the disk.

To get a Dirac fermion on the boundary, we will need two flavors of Dirac on the disk.

Each has its own exact $U(1)$ conserved currents.

Seems to lead to exactly conserved $U(1)$ currents for both vector and axial symmetry.

Is this a bug or a feature?

What could go wrong?

For QCD as the target theory:

If the $U(1)$ axial symmetry is exact, instead of a massive η' you may end up with a Goldstone mode instead.

Undesirable.

Golterman-Shamir: Phys.Rev.D 109 (2024) 11, 114519

What happens in the domain wall construction for the disk, then?

Physics of η' in 1-flavor QCD

With an exact $U(1)_A$ symmetry that is broken by a small quark mass:

$$L = \frac{1}{2}\partial_\mu \eta' \partial_\mu \eta' - M_{\eta,Q}^2 \cos\left(\frac{\eta'}{f} - \theta\right) + \ \ldots$$

shift symmetry of the Goldstone mode

shift symmetry breaking, mass of η'

QCD θ angle, which can be rotated away without the anomaly

Instantons and chiral anomaly

Gauge field configs with nonzero windings host fermion zero modes:

Produces $\bar{q}_R q_L$ vertices.

Sum over all possible positions of instantons and anti-instantons exponentiates the effective vertex ('t Hooft's instanton gas model)

$$e^{-S} = \sum_{n,\bar{n}} \frac{\left(\Lambda \int dx\, \bar{q}_R q_L\right)^n}{n!} \frac{\left(\Lambda \int dy\, \bar{q}_L q_R\right)^{\bar{n}}}{\bar{n}!} = e^{\Lambda \int dx\, (\bar{q}_R q_L + \bar{q}_L q_R)}$$

QCD scale (not to be determined in the model)

η' effective field theory

Chiral condensate

$$\bar{q}_R q_L \rightarrow \frac{\Sigma}{2} e^{\frac{i\eta'}{f}} \qquad \bar{q}_L q_R \rightarrow \frac{\Sigma}{2} e^{-\frac{i\eta'}{f}} \qquad \frac{\Lambda\Sigma}{f^2} = M_{\eta'}^2$$

Decay constant

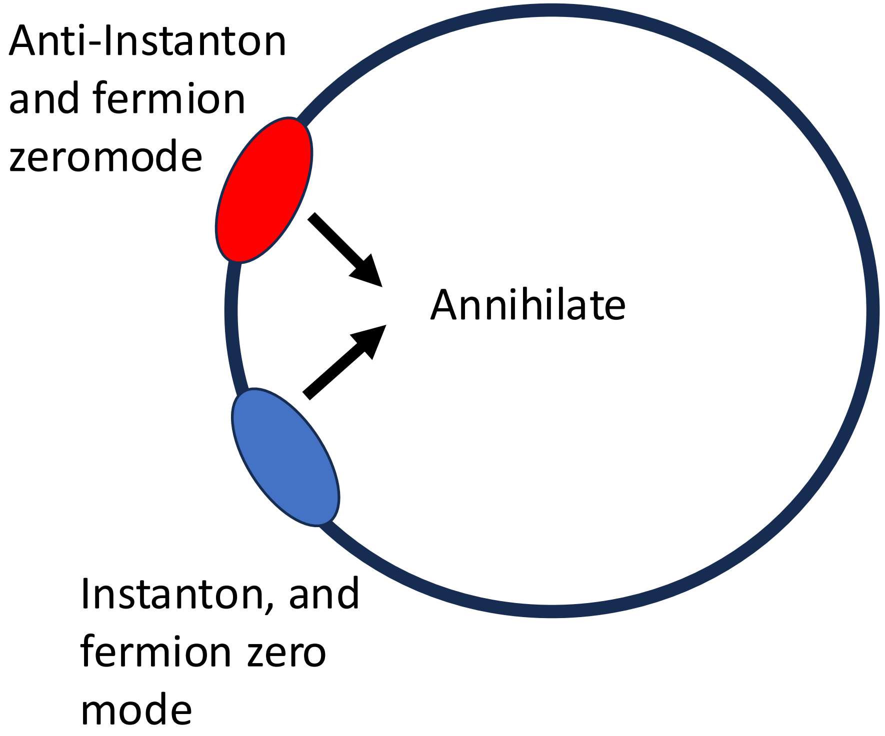
$$L = \frac{1}{2}\left(\partial_\mu \eta'\right)^2 - {M'_\eta}^2 f^2 \cos\frac{\eta'}{f} - \frac{\mathrm{M_q}\Sigma}{f^2} \cos\left(\frac{\eta'}{f} - \theta\right)$$

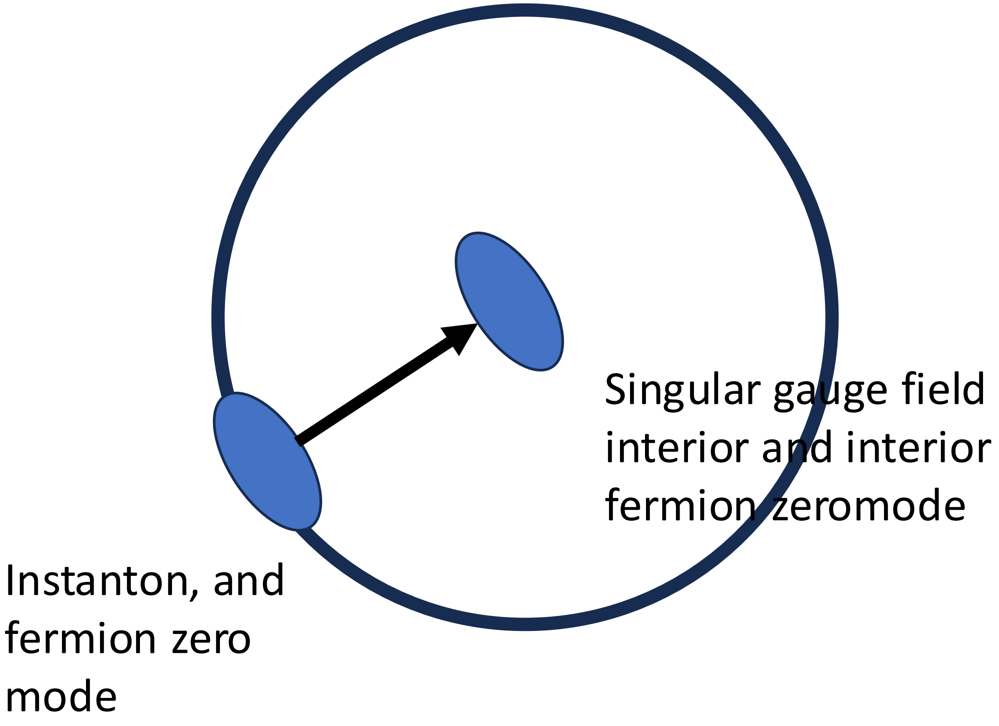
Instanton/anomaly contribution

Quark mass contribution

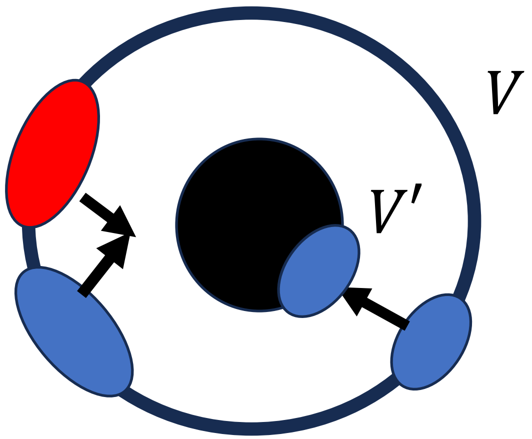
Bulk of η' mass is coming from the anomaly/instanton contribution and θ cannot be rotated away -> Strong CP problem

What happens on the disk? (flow equation)





Modification to ‘t Hooft’s result



q = outer boundary zeromode   Q= inner boundary zeromode

$$\mathcal{O} = \Lambda \int_V \frac{d^4x}{V}\, \bar{q}_R q_L\ , \quad \bar{\mathcal{O}} = \Lambda \int_V \frac{d^4x}{V}\, \bar{q}_L q_R\ , \quad X = \Lambda \int_{V'} \frac{d^4y}{V'}\, \bar{Q}_R Q_L\ , \quad \bar{X} = \Lambda \int_{V'} \frac{d^4y}{V'}\, \bar{Q}_L Q_R\ ,$$

$$\begin{aligned} e^{-\tilde{S}_{\text{int}}} &= \sum_{n,\bar{n}} e^{i(n-\bar{n})\theta} \frac{(V\mathcal{O})^n}{n!} \frac{(V\bar{\mathcal{O}})^{\bar{n}}}{\bar{n}!} \left((V'X)^{(n-\bar{n})} \Theta(n-\bar{n}) + (V'\bar{X})^{(\bar{n}-n)} \Theta(\bar{n}-n) + \underline{\delta_{n,\bar{n}}} \right) \\ &= Z_1 + Z_2 + Z_3 \end{aligned}$$

$$\Theta(n) = \begin{cases} 1 & n > 0 \\ 0 & n \leq 0 \end{cases}$$

This is the operator we have access to in our world

Modification to ‘t Hooft’s result

$$Z_3 \equiv e^{-\widetilde{S}_{\text{inst.}}} = \sum_{n=0}^{\infty} \frac{\left(\Lambda \int \bar{q}_R q_L(x)\, dx\right)^n}{n!} \frac{\left(\Lambda \int \bar{q}_L q_R(y)\, dy\right)^n}{n!} = I_0\left(2\Lambda V \sqrt{\overline{\bar{q}_R q_L}\ \overline{\bar{q}_L q_R}}\right)$$

Bessel function

Assuming chiral symmetry breaking, the η' Lagrangian

For $\quad V \to \infty \qquad \tilde{S}_{\text{inst}} \to -V\left(2\Lambda \sqrt{\overline{\bar{q}_R q_L}\quad \overline{\bar{q}_L\, q_R}}\ \right)$

Bar notation = spatial average:

$$\overline{\mathcal{O}} \equiv \int \frac{d^4x}{V}\, \mathcal{O}(x)$$

$$S_{\text{inst.}} = \int dx\, \tfrac{1}{2}\partial_\mu \eta' \partial_\mu \eta' - V M_{\eta'}^2 f^2 \sqrt{\overline{\left(e^{i\eta'/f}\right)}\ \overline{\left(e^{-i\eta'/f}\right)}}\,.$$

$$\mathcal{L}_{\text{inst.}} = \tfrac{1}{2}\partial_\mu \eta' \partial_\mu \eta'(x) + M_{\eta'}^2 f^2 \left[-1 + \tfrac{1}{2}\left(\eta'(x) - \overline{\eta'}\right)^2 + O(\eta'^4)\right]$$

Strong CP

$$\mathcal{L}_{\text{inst.}} = \frac{1}{2}\partial_\mu\eta'\partial_\mu\eta'(x) + M_{\eta'}^2 f^2\left[-1 + \frac{1}{2}\left(\eta'(x) - \overline{\eta'}\right)^2 + O(\eta'^4)\right]$$

We have a mass for η'.
However, every other mode except the $p = 0$ mode gets a mass.

So, the action respects the shift symmetry, indicative of the exact U(1) symmetry

And you can rotate away the theta angle by rotating the uniform mode!

What this means

Absence of strong CP without an axion. It's the η' which is absorbing the θ angle. Yet, η' is massive here.

Thus, we started out with finding a lattice regulator for chiral gauge theories and have ended up not only with a proposal for CGT, but also a proposal for eliminating the strong CP problem.

Remarkably, this can be tested!

The test and future direction

In general, hard to test a proposal for lattice chiral gauge theories since the interesting ones have a sign problem.

But, our proposal comes with a test in the QCD sector of the standard model. So, the test is to simulate 1-flavor QCD using the disk setup.

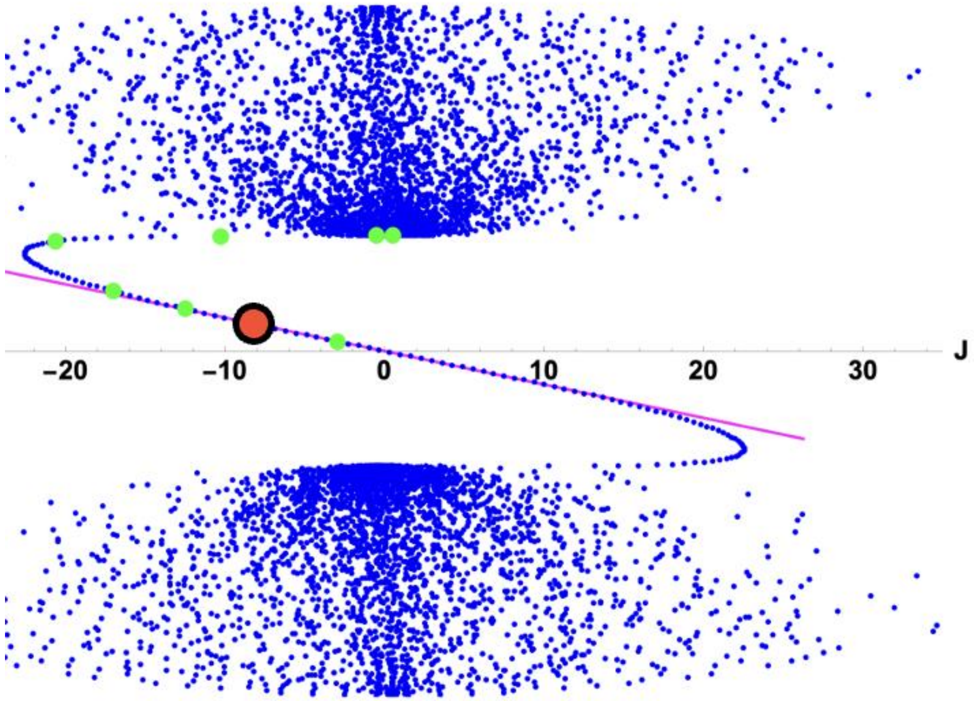
Extract the η' spectra and the $p = 0$ mode.

Will be a check for the proposal as well as evidence that the θ angle can be removed!

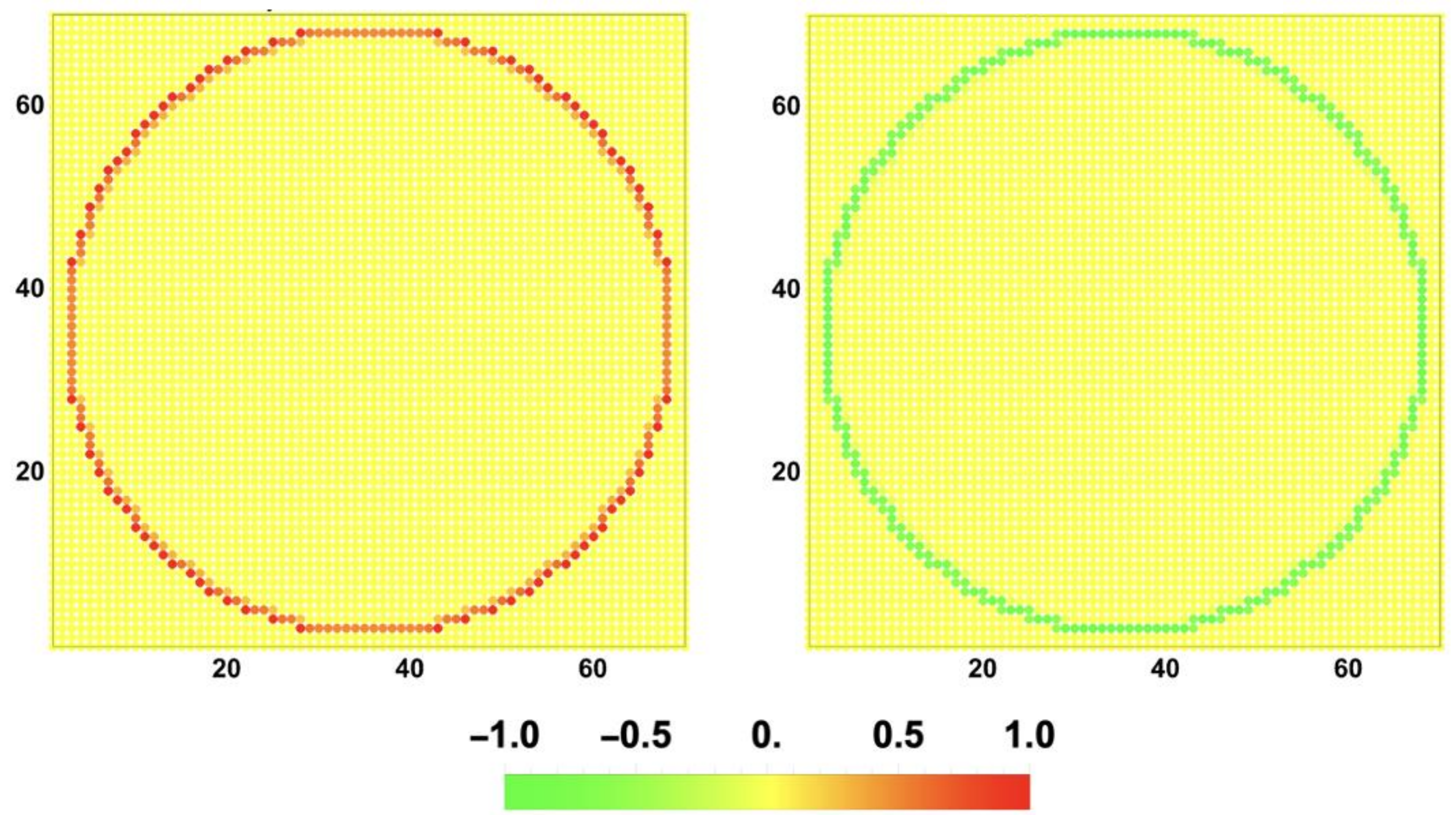
Future work

- Make sense of this construction in Minkowski space?
- Sign problem in Euclidean formulation!
- Can we find a Hamiltonian formulation? With a potential quantum simulation in mind in the future?
- If our world is a boundary of a five-dimensional world, are there any consequences?

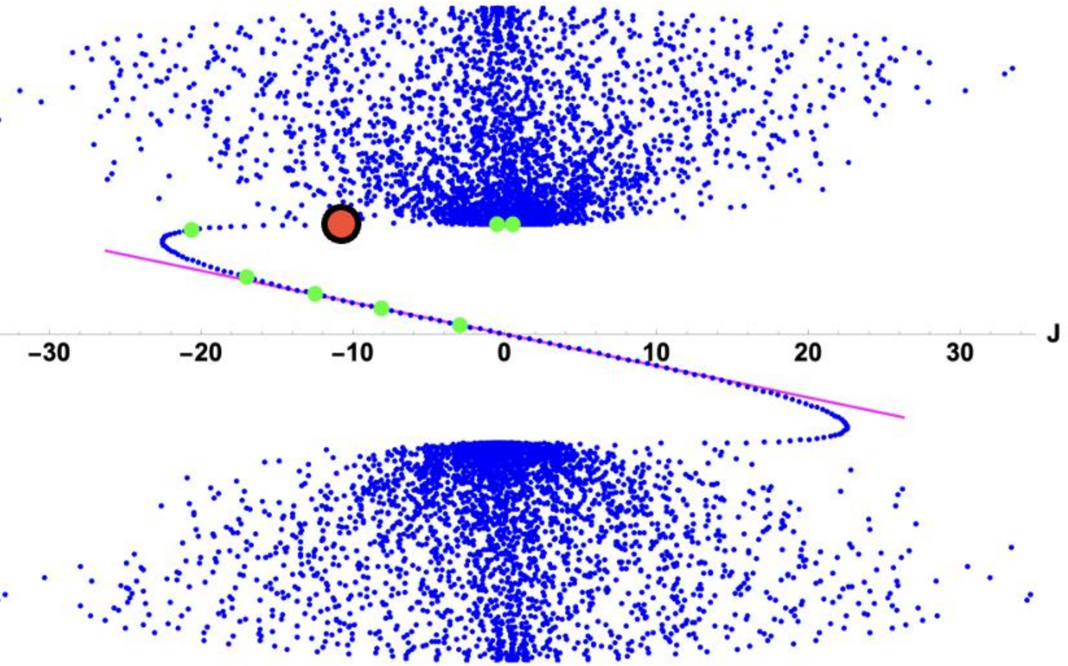
Thank you

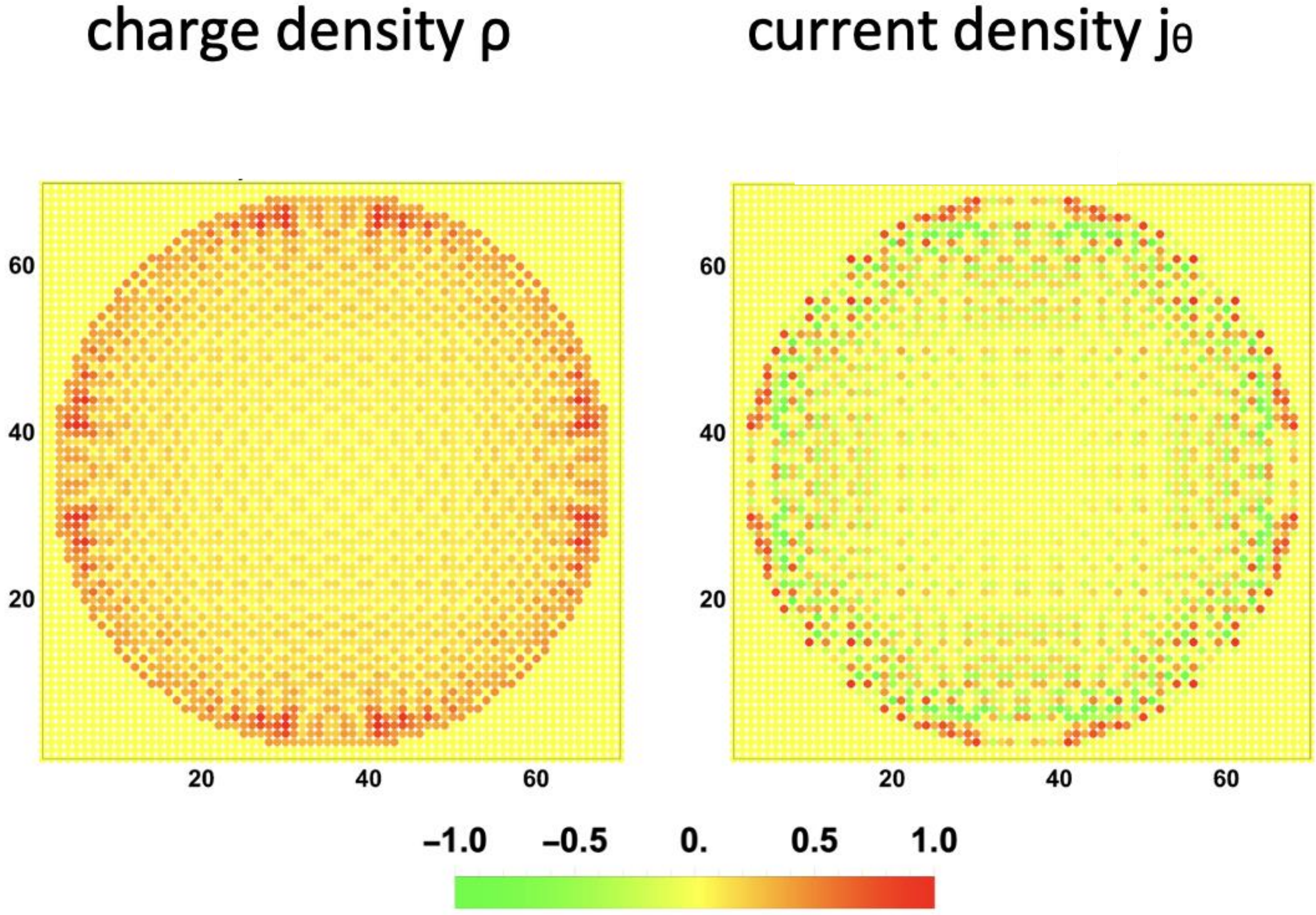


charge density ρ current density j_θ



Kaplan, Sen, *Phys.Rev.Lett.* 132 (2024) 14, 141604





Kaplan, Sen, *Phys.Rev.Lett.* 132 (2024) 14, 141604