Exactly Solvable 1+1d Chiral Lattice Gauge Theories

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to appear with **Yichul Choi**, **Nathan Seiberg**, and **Wucheng Zhang**

Lattice formulation of the Standard Model

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It is hard to put *chiral fermions* on the lattice due to

1. The Nielsen-Ninomiya theorem
2. Anomalies of chiral fermions

These obstacles make it hard, but not impossible, to find a lattice regularization of the Standard Model!

- The Standard Model is anomaly-free
- The goal is anomaly-free chiral lattice gauge theories
  [..., Wang-Wen 13, Kaplan-Sen 23, Butt-Catterall-Hasenfratz 24, ...]
- We address this in **1+1d** for **abelian** gauge theories
  See also [Berkowitz-Cherman-Jacobson 23, Thorngren-Preskill-Fidkowski 26]

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- Lattice regularization requires the realization of exact chiral symmetries of individual chiral fermions on the lattice
- The IR theory may have emergent parity symmetry
- The UV theory may have a hidden parity symmetry and be non-chiral!

Evading the Nielsen-Ninomiya no-go theorem

There is no discretization of the Dirac operator $D(p)$, such that

- **Locality**: $D(p)$ is a continuous and periodic function of p
- **Free fermions**: $D(p) \sim \gamma_\mu p^\mu$ for $|p| \ll 1/a$
- **No doublers**: $D(p)$ has poles only at $p = 0$
- **Exact chiral symmetry:** $\{\Gamma, D(p)\} = 0$

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Our strategy is to use duality/bosonization instead of directly discretizing the Dirac operators on the lattice: [Neuzil's talk]

1. Realizing exact anomalous chiral symmetries on the lattice
   Modified Villain: [Gross-Klebanov 90, Sulejmanpasic-Gattringer 19, Gorantla-Lam-Seiberg-Shao 21, Cheng-Seiberg 22, Fazza-Sulejmanpasic 22, Xu-Chen 24, Fidkowski-Xu-Zhang 25]
2. Gauging anomaly-free combination of chiral symmetries
   [SS 2023]

Goal: 1+1d anomaly-free U(1) gauge theories

I. Realizing N free Dirac fermions and their $\mathrm{U}(1)_{\mathsf{L}}^{N} \times \mathrm{U}(1)_{\mathsf{R}}^{N}$ chiral symmetry on the lattice

$$\psi_{\mathsf{L}}^{(1)}, \psi_{\mathsf{L}}^{(2)}, \cdots, \psi_{\mathsf{L}}^{(N)} \quad , \quad \psi_{\mathsf{R}}^{(1)}, \psi_{\mathsf{R}}^{(2)}, \cdots, \psi_{\mathsf{R}}^{(N)}$$

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II. Gauge an anomaly-free U(1) symmetry where the left-moving and right-moving fermions have integer charges

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- No gravitational anomaly: $c_\mathsf{L} = c_\mathsf{R} = N$
- No gauge anomaly if:

$$\sum_{I=1}^{N} (n_\mathsf{L}^{(I)})^2 = \sum_{I=1}^{N} (n_\mathsf{R}^{(I)})^2$$

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- One popular option is the "34-50" model: $3^2 + 4^2 = 5^2 + 0^2$

Outline

1. Introduction
2. The continuum theory
3. Modified Villain formulation of compact boson CFT
4. Lattice gauge theories
   I. The Schwinger model
   II. The "34-50" model

Bosonization: Dirac fermion $\Leftrightarrow$ compact boson

A Dirac fermion in 1+1d is related by bosonization to the compact boson theory at radius $R = \frac{1}{\sqrt{2}}$:

$$\mathcal{L}_{\mathsf{Dirac}} = i\psi_{\mathsf{L}}^{\dagger}\partial_{-}\psi_{\mathsf{L}} + i\psi_{\mathsf{R}}^{\dagger}\partial_{+}\psi_{\mathsf{R}} \quad \Longleftrightarrow \quad \mathcal{L}_{\mathsf{boson}} = \frac{1}{8\pi}\partial_{\mu}\varphi\partial^{\mu}\varphi$$

where $\varphi \sim \varphi + 2\pi$. The compact boson theory has $\mathrm{U}(1)_{\mathsf{m}} \times \mathrm{U}(1)_{\mathsf{w}}$ symmetry, which is related to chiral symmetries of fermions as

$$Q_{\mathsf{L}} = Q_{\mathsf{m}} + \frac{1}{2}Q_{\mathsf{w}} \qquad Q_{\mathsf{R}} = Q_{\mathsf{m}} - \frac{1}{2}Q_{\mathsf{w}}$$

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- In the "34-50" model, the anomaly-free conserved charge is:

$$Q = 3Q_{\text{L}}^{(1)} + 4Q_{\text{L}}^{(2)} + 5Q_{\text{R}}^{(1)} = 8Q_{\text{m}}^{(1)} + 4Q_{\text{m}}^{(2)} - Q_{\text{w}}^{(1)} + 2Q_{\text{w}}^{(2)}$$

N compact bosons

Consider N compact bosons $\varphi^{(I)} \sim \varphi^{(I)} + 2\pi$ (for $I = 1, 2, \cdots, N$) with moduli $E = G + B$:

$$S = \frac{1}{4\pi} \int \left(G_{IJ}\, \mathrm{d}\varphi^{(I)} \wedge \star \mathrm{d}\varphi^{(J)} + B_{IJ}\, \mathrm{d}\varphi^{(I)} \wedge \mathrm{d}\varphi^{(J)} \right)$$

The momentum and winding currents

$$J_{\mathrm{m}}^{(I)} = \frac{1}{2\pi} \left(G_{IJ} \star \mathrm{d}\varphi^{(J)} + B_{IJ}\, \mathrm{d}\varphi^{(J)} \right) \,, \qquad J_{\mathrm{w}}^{(I)} = \frac{1}{2\pi} \mathrm{d}\varphi^{(I)}$$

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- There is a mixed 't Hooft anomaly between momentum and winding symmetries

$$\alpha_{\mathsf{anomaly}} = -\frac{i}{2\pi} \sum_{I=1}^{N} \int A_{\mathsf{w}}^{(I)} \wedge \mathrm{d}A_{\mathsf{m}}^{(I)}$$

Anomaly cancellation: N compact bosons

$$\alpha[A_{\mathrm{m}}^{(I)}, A_{\mathrm{w}}^{(J)}] = -\frac{i}{2\pi} \sum_{I=1}^{N} \int A_{\mathrm{w}}^{(I)} \wedge \mathrm{d}A_{\mathrm{m}}^{(I)}$$

For the U(1) symmetry generated by the conserved charge

$$Q = \sum_{I=1}^{N} \left(n_{\mathrm{m}}^{(I)} Q_{\mathrm{m}}^{(I)} + n_{\mathrm{w}}^{(I)} Q_{\mathrm{w}}^{(I)} \right)$$

the anomaly is given by setting $A_{\mathrm{m}}^{(I)} = n_{\mathrm{m}}^{(I)} A$ and $A_{\mathrm{w}}^{(I)} = n_{\mathrm{w}}^{(I)} A$:

$$\alpha[A] = -\left(\sum_{I=1}^{N} n_{\mathrm{m}}^{(I)} n_{\mathrm{w}}^{(I)} \right) \int \frac{2}{4\pi} A \mathrm{d}A$$

$$\boxed{\text{Anomaly cancellation:} \quad \vec{n}_{\mathrm{m}} \cdot \vec{n}_{\mathrm{w}} = 0}$$

T-duality

A single compact boson at radius R is equivalent to one at $\tilde{R} = 1/R$, and T-duality exchanges momentum and winding:

$$J_{\mathsf{m}} = \frac{R^2}{2\pi} \star \mathrm{d}\varphi = \frac{1}{2\pi}\mathrm{d}\tilde{\varphi} \quad \longleftrightarrow \quad J_{\mathsf{w}} = \frac{1}{2\pi}\mathrm{d}\varphi = \frac{\tilde{R}^2}{2\pi} \star \mathrm{d}\tilde{\varphi}$$

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For N compact bosons, the T-duality group is $\mathrm{O}(N, N; \mathbb{Z})$, generated by

1. T-dualities of individual bosons
2. $\mathrm{GL}(N, \mathbb{Z})$ basis changes of the fields $\varphi^{(I)}$
3. Integer shifts of B_{IJ}

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which acts on the momentum/winding charges as:

$$\begin{pmatrix} \vec{n}_{\mathrm{m}} \\ \vec{n}_{\mathrm{w}} \end{pmatrix} \longmapsto \mathcal{O} \begin{pmatrix} \vec{n}_{\mathrm{m}} \\ \vec{n}_{\mathrm{w}} \end{pmatrix} , \qquad \mathcal{O} \in \mathrm{O}(N, N; \mathbb{Z})$$

Solving the gauge theory via T-duality

Strategy: use $\mathcal{O} \in \mathrm{O}(N, N; \mathbb{Z})$ to rotate the vector of charges $(\vec{n}_{\mathrm{m}}, \vec{n}_{\mathrm{w}})$ so that only the winding of the first boson is gauged:

$$\mathcal{O} \begin{pmatrix} \vec{n}_{\mathrm{m}} \\ \vec{n}_{\mathrm{w}} \end{pmatrix} = n \begin{pmatrix} \vec{0} \\ \vec{e}_1 \end{pmatrix} \ , \qquad n = \mathrm{gcd}(\vec{n}_{\mathrm{m}}, \vec{n}_{\mathrm{w}})$$

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In this frame, the gauged charge is non-chiral (winding of $\varphi^{(1)}$). The chirality is hidden in the moduli of the remaining bosons:

$$S = \int \frac{1}{2e^2} \mathrm{d}a \wedge \star \mathrm{d}a + \frac{G'_{IJ}}{4\pi} \mathrm{d}\varphi^{(I)} \wedge \star \mathrm{d}\varphi^{(J)} + \frac{B'_{IJ}}{4\pi} \mathrm{d}\varphi^{(I)} \wedge \mathrm{d}\varphi^{(J)} + \frac{n}{2\pi} \varphi^{(1)} \, \mathrm{d}a$$

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- This theory can be solved exactly by integrating out the gauge field 'a', which makes $\varphi^{(1)}$ noncompact and massive
- The theory factorizes as: a non-compact boson $\otimes$ $(N-1)$ compact bosons $\otimes$ a $\mathbb{Z}_n$ gauge theory

Modified Villain model: $\mathcal{L} = \frac{R^2}{4\pi}((\partial_0\varphi)^2 + (\partial_1\varphi)^2)$

The idea is to start from the non-compact boson theory

$$H = \sum_{j=1}^{L} \left(\frac{1}{2} p_j^2 + \frac{1}{2}(\phi_{j+1} - \phi_j)^2 \right) \quad \text{where: } [\phi_j, p_{j'}] = i\delta_{j,j'}$$

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and gauge the shift symmetry $\phi_j \mapsto \phi_j + 2\pi R\mathbb{Z}$ to find

$$H_{\mathrm{mV}} = \sum_{j=1}^{L} \left[\frac{1}{2R^2} p_j^2 + \frac{R^2}{2} \left(\frac{\phi_{j+1} - \phi_j}{2\pi} + \tilde{p}_{j,j+1} \right)^2 \right]$$

where $\tilde{p}_{j,j+1}$ are integer-valued gauge fields. They satisfy $[\tilde{\phi}_{j,j+1}, \tilde{p}_{j',j'+1}] = i\delta_{j,j'}$ and Gauss's law constraints:

$$e^{2\pi i p_j - i(\tilde{\phi}_{j,j+1} - \tilde{\phi}_{j-1,j})} = 1 : \qquad \begin{array}{l} \phi_j \sim \phi_j + 2\pi n_j \\ \tilde{p}_{j,j+1} \sim \tilde{p}_{j,j+1} + n_j - n_{j+1} \end{array}$$

$$e^{2\pi i \tilde{p}_{j,j+1}} = 1 : \qquad \tilde{\phi}_{j,j+1} \sim \tilde{\phi}_{j,j+1} + 2\pi m_{j,j+1}$$

The chiral global symmetries of H_{mV}

$$Q_{\mathsf{m}} = \sum_{j=1}^{L} p_j \ , \qquad Q_{\mathsf{w}} = \sum_{j=1}^{L} \left(\tilde{p}_{j,j+1} + \frac{\phi_{j+1} - \phi_j}{2\pi} \right)$$

The momentum/winding charges are manifestly gauge invariant.
They are also quantized because of the gauge constraints:
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- The anomaly can be seen from the Schwinger term:

$$\left[p_j \ , \ \left(\tilde{p}_{j',j'+1} + \frac{\phi_{j'+1} - \phi_{j'}}{2\pi} \right) \right] = \frac{i}{2\pi} \left(\delta_{j,j'} - \delta_{j,j'+1} \right)$$

which is a discretization of $\left[J^0_{\mathrm{m}}(t,x), J^0_{\mathrm{w}}(t,y) \right] = \frac{i}{2\pi} \partial_x \delta(x-y)$

The chiral global symmetries of $H_{\rm mV}$

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- Different from the chiral anomaly in [Chatterjee-Pace-Shao 24, Catterall-Pradhan-Samlodia 25, Lew-Smith-Pace-Shao 26]

T-duality of H_{mV}

The modified Villain Hamiltonian realizes an exact T-duality

$$\mathcal{T}: \quad \begin{array}{ll} \phi_j \mapsto \tilde{\phi}_{j,j+1} \ , & p_j \mapsto \tilde{p}_{j,j+1} + \dfrac{\phi_{j+1} - \phi_j}{2\pi} \\ \tilde{\phi}_{j,j+1} \mapsto \phi_{j+1} \ , & \tilde{p}_{j,j+1} \mapsto p_{j+1} - \dfrac{\tilde{\phi}_{j+1,j+2} - \tilde{\phi}_{j,j+1}}{2\pi} \end{array}$$

It implements T-duality and is gauge invariant since

$$\mathcal{T}: \quad \begin{array}{l} H_{\mathrm{mV}}\big(R\big) \mapsto H_{\mathrm{mV}}\big(R^{-1}\big) \\ e^{2\pi i p_j - i(\tilde{\phi}_{j,j+1} - \tilde{\phi}_{j-1,j})} \mapsto e^{2\pi i \tilde{p}_{j,j+1}} \mapsto e^{2\pi i p_{j+1} - i(\tilde{\phi}_{j+1,j+2} - \tilde{\phi}_{j,j+1})} \end{array}$$

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- $\mathcal{T}$ exchanges the momentum and winding charges as expected: $\mathcal{T} Q_{\mathrm{m}} \mathcal{T}^{-1} = Q_{\mathrm{w}}$ and $\mathcal{T} Q_{\mathrm{w}} \mathcal{T}^{-1} = Q_{\mathrm{m}}$
- $\mathcal{T}$ acts as a half-translation since $\mathcal{T}^2 = T_{\mathrm{translation}}$

1. Introduction

2. The continuum theory

3. Modified Villain formulation of compact boson CFT

4. Lattice gauge theories

I. The Schwinger model

II. The "34-50" model

Constructing chiral lattice gauge theories

We start with N flavors of the modified Villain theory

$$H_{\mathrm{mV}} = \sum_{I=1}^{N} \sum_{j=1}^{L} \left[\frac{1}{2R^2} \left(p_j^{(I)} \right)^2 + \frac{R^2}{2} \left(\tilde{p}_{j,j+1}^{(I)} + \frac{\phi_{j+1}^{(I)} - \phi_j^{(I)}}{2\pi} \right)^2 \right]$$

with constraints $e^{2\pi i p_j^{(I)} - i(\tilde{\phi}_{j,j+1}^{(I)} - \tilde{\phi}_{j-1,j}^{(I)})} = 1$ and $e^{2\pi i \tilde{p}_{j,j+1}^{(I)}} = 1$

- Consider an anomaly-free U(1) chiral symmetry associated with the conserved charge

$$Q = \sum_{I=1}^{N} (n_{\mathrm{m}}^{(I)} Q_{\mathrm{m}}^{(I)} + n_{\mathrm{w}}^{(I)} Q_{\mathrm{w}}^{(I)}) \ , \qquad \text{where:} \ \sum_{I=1}^{N} n_{\mathrm{m}}^{(I)} n_{\mathrm{w}}^{(I)} = 0$$

- Gauge this symmetry following [SS 2023]

Gauging $\mathrm{U}(1)_Q$

I. We couple the theory to background U(1) gauge fields $a_j \sim a_j + 2\pi$. This amounts to modifying Gauss's laws to

$$e^{2\pi i p_j^{(I)} - i(\tilde{\phi}^{(I)}_{j,j+1} - \tilde{\phi}^{(I)}_{j-1,j})} = e^{i n_{\mathrm{w}}^{(I)} a_j} \,, \qquad e^{2\pi i \tilde{p}^{(I)}_{j-1,j}} = e^{i n_{\mathrm{m}}^{(I)} a_j}$$

II. Making the gauge fields dynamical. We add conjugate variables (the electric field) E_j, where $[a_j, E_{j'}] = i\delta_{j,j'}$ and impose Gauss's law constraints

$$\sum_{I=1}^{N} \left(n_{\mathrm{m}}^{(I)} q_j^{(I)} + n_{\mathrm{w}}^{(I)} q_{j,j+1}^{(I)} \right) = E_j - E_{j+1} \,, \quad e^{2\pi i E_j} = 1$$

III. Add a kinetic term for the gauge fields to the Hamiltonian

$$\frac{e^2}{2} \sum_{j=1}^{L} \left(E_j + \frac{\theta}{2\pi} - \sum_{I=1}^{N} \frac{n_{\mathrm{w}}^{(I)} \phi_j^{(I)} + n_{\mathrm{m}}^{(I)} \tilde{\phi}^{(I)}_{j-1,j}}{2\pi} \right)^2$$

$N=1$ case: the "massless" Schwinger model

$$H_{\text{Schwinger}} = \sum_{j=1}^{L} \frac{\mathsf{e}^2\mathsf{a}}{2}\left(E_j + \frac{\theta}{2\pi} - \frac{\phi_j}{2\pi}\right)^2 + \sum_{j=1}^{L} \frac{2\pi}{\mathsf{a}} p_j^2 + \sum_{j=1}^{L} \frac{2\pi}{4\mathsf{a}}\left(\tilde{p}_{j,j+1} + \frac{\phi_{j+1}-\phi_j}{2\pi}\right)^2$$

with constraints

$$e^{2\pi i p_j - i(\tilde{\phi}_{j,j+1} - \tilde{\phi}_{j-1,j})} = e^{i a_j} \,, \qquad e^{2\pi i \tilde{p}_{j,j+1}} = 1$$

$$\tilde{p}_{j,j+1} + E_{j+1} - E_j = 0 \,, \qquad e^{2\pi i E_j} = 1$$

N=1 case: the "massless" Schwinger model

$$H_{\text{Schwinger}} = \sum_{j=1}^{L} \frac{e^2 a}{2}\left(\cancel{E_j} + \frac{\theta}{2\pi} - \frac{\phi_j}{2\pi}\right)^2 + \sum_{j=1}^{L} \frac{2\pi}{a} p_j^2$$

$$+ \sum_{j=1}^{L} \frac{2\pi}{4a}\left(\cancel{\tilde{p}_{j,j+1}} + \frac{\phi_{j+1} - \phi_j}{2\pi}\right)^2$$

with constraints

$$\cancel{e^{2\pi i p_j - i(\tilde{\phi}_{j,j+1} - \tilde{\phi}_{j-1,j})}} = e^{i a_j}\ , \qquad e^{2\pi i \tilde{p}_{j,j+1}} = e^{2\pi i (E_{j+1} - E_j)}$$

$$\tilde{p}_{j,j+1} + \cancel{E_{j+1} - E_j} = 0\ , \qquad e^{2\pi i E_j} = 1$$

- We apply the unitary transformation $e^{2\pi i \sum_{j=1}^{L} E_j q_j}$

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- We apply the unitary transformation $e^{2\pi i \sum_{j=1}^{L} E_j q_j}$ to get

$$H_{\text{Schwinger}} = \sum_{j=1}^{L} \frac{\mathrm{e}^2 \mathrm{a}}{8\pi^2} \phi_j^2 + \sum_{j=1}^{L} \frac{2\pi}{\mathrm{a}} p_j^2 + \sum_{j=1}^{L} \frac{2\pi}{4\mathrm{a}} \left(\frac{\phi_{j+1} - \phi_j}{2\pi} \right)^2$$

which describes a non-compact boson of mass $M = \frac{\mathrm{e}}{\sqrt{\pi}}$

The "34-50"/(8,4,-1,2) model

$$H_{8,4,-1,2} = \frac{e^2 a}{2} \sum_{j=1}^{L} \left(E_j + \frac{\theta}{2\pi} - \frac{8\tilde{\phi}^{(1)}_{j-1,j} + 4\tilde{\phi}^{(2)}_{j-1,j} - \phi^{(1)}_j + 2\phi^{(2)}_j}{2\pi} \right)^2$$

$$+ \frac{2\pi}{a} \sum_{j=1}^{L} \sum_{I=1}^{2} \left(\left(p^{(I)}_j \right)^2 + \frac{1}{4} \left(\tilde{p}^{(I)}_{j,j+1} + \frac{\phi^{(I)}_{j+1} - \phi^{(I)}_j}{2\pi} \right)^2 \right)$$

with constraints

$$e^{2\pi i q^{(1)}_j} = e^{-i a_j} \;, \qquad e^{2\pi i q^{(1)}_{j-1,j}} = e^{8 i a_j}$$

$$e^{2\pi i q^{(2)}_j} = e^{2 i a_j} \;, \qquad e^{2\pi i q^{(2)}_{j-1,j}} = e^{4 i a_j}$$

$$e^{2\pi i E_j} = 1 \;, \qquad 8q^{(1)}_j + 4q^{(2)}_j - q^{(1)}_{j,j+1} + 2q^{(2)}_{j,j+1} = E_j - E_{j+1}$$

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- To solve this, we perform a T-duality transformation $\mathcal{T}_{\mathcal{M}}$

The fermionic 34-50 model

The Hilbert space consists of two bosons $\phi_j^{(1)}, \phi_j^{(2)}$ and two Majorana fermions $\psi_j, \tilde{\psi}_j$ per site and one boson $\tilde{\phi}^{(2)}_{j,j+1}$ per link:

$$H_{3450} = \frac{e^2}{8\pi^2} \sum_{j=1}^{L} \left(\phi_j^{(1)}\right)^2 + \sum_{j=1}^{L} \left(\left(p_j^{(1)}\right)^2 + \left(-p_j^{(2)} + 2p_j^{(1)}\right)^2 \right)$$

$$+ \sum_{j=1}^{L} \frac{1}{4} \left(-2\frac{(\mathrm{D}\phi^{(2)})_{j,j+1}}{2\pi} - 4p_j^{(2)} + 8p_{j+1}^{(1)} - \frac{(d\phi^{(1)})_{j,j+1}}{2\pi} \right)^2$$

$$+ \sum_{j=1}^{L} \frac{1}{4} \left(-\tilde{p}^{(2)}_{j,j+1} - \frac{\phi^{(2)}_{j+1} - \phi^{(2)}_j}{2\pi} + 4p^{(1)}_{j+1} \right)^2$$

with Gauss's law constraints

$$e^{2\pi i p_j^{(2)} - i(\tilde{\phi}^{(2)}_{j,j+1} - \tilde{\phi}^{(2)}_{j-1,j})} = i\tilde{\psi}_j \psi_j \ , \qquad i\psi_j \, e^{i\pi \tilde{p}^{(2)}_{j,j+1}} \, \tilde{\psi}_{j+1} = 1$$

The solution

The modified Villain theory has an exact $\mathrm{O}(N, N; \mathbb{Z})$ T-duality.
Using this, we solve the "34-50" model and determine its spectrum:
a massless Dirac fermion + a Schwinger boson of mass $M = 5e/\sqrt{\pi}$ [Choi-Seiberg-SS-Zhang]

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- The "34-50" model has a hidden reflection symmetry that is not manifest in the fermionic d.o.f. It is actually non-chiral!
- To get a chiral gauge theory, we need $N \geq 3$ Dirac fermions
- We can break the exact solvability by adding mass terms for the fermions (e.g., two-flavor massive Schwinger model [Cuomo-Dempsey-Katsevich-Klebanov-Kochergin-Pufu-Søgaard '26])

Summary & Outlook

- **1+1d:** Constructed exactly solvable chiral lattice $\mathrm{U}(1)$ gauge theories via gauging anomaly-free combinations of chiral symmetries

- Modified Villain model realizes all the moduli of N compact bosons and their full $\mathrm{O}(N, N; \mathbb{Z})$ T-duality on the lattice

- **3+1d:** There is a lattice realization of exact $\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}$ chiral symmetry and its anomaly [Thorngren, Preskill, Fidkowski 26] in a concrete lattice Hamiltonian of bosons [Lu, SS, Shao 26]

- Generalization to **non-abelian** gauge groups such as SU(N)?

- Chiral symmetry in 3+1d with U(1)-**grav-grav** anomaly?

Thank You!

Alternative presentation of the gauge theory

Applying the transformation $e^{-i\sum_{I,j}(\frac{n_{\mathrm{w}}^{(I)}}{2\pi}\phi_j^{(I)}+\frac{n_{\mathrm{m}}^{(I)}}{2\pi}\tilde{\phi}_{j-1,j}^{(I)})a_j}$, the gauged Hamiltonian becomes

$$H_{\mathrm{gauged}}=\frac{e^2}{2}\sum_{j=1}^{L}\left(E_j+\frac{\theta}{2\pi}\right)^2+\sum_{I=1}^{N}\sum_{j=1}^{L}\frac{1}{2R^2}\left(p_j^{(I)}+\frac{n_{\mathrm{w}}^{(I)}}{2\pi}a_j\right)^2$$

$$+\sum_{I=1}^{N}\sum_{j=1}^{L}\frac{R^2}{2}\left(\tilde{p}_{j,j+1}^{(I)}+\frac{n_{\mathrm{m}}^{(I)}}{2\pi}a_{j+1}+\frac{\phi_{j+1}^{(I)}-\phi_j^{(I)}}{2\pi}\right)^2$$

Gauss's law for U(1) gauging and the quantization of the electric fields change to

$$\sum_{I=1}^{N}\left(n_{\mathrm{m}}^{(I)}p_j^{(I)}+n_{\mathrm{w}}^{(I)}\left(\tilde{p}_{j,j+1}^{(I)}+\frac{\phi_{j+1}^{(I)}-\phi_j^{(I)}}{2\pi}\right)\right)=E_j-E_{j+1}$$

$$e^{2\pi i E_j}=\exp\left\{-i\sum_{I=1}^{N}\frac{n_{\mathrm{w}}^{(I)}}{2\pi}\phi_j^{(I)}+\frac{n_{\mathrm{m}}^{(I)}}{2\pi}\tilde{\phi}_{j-1,j}^{(I)}\right\}$$

Solving the (8,4,-1,2) model using T-duality

$$\mathcal{M} = \begin{pmatrix} -1 & 2 & 8 & 4 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & -1 \end{pmatrix} : \begin{pmatrix} 8 \\ 4 \\ -1 \\ 2 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathcal{T}_{\mathcal{M}} Q \mathcal{T}_{\mathcal{M}}^{-1} = Q_{\mathrm{w}}^{(1)}$$

$$\mathcal{T}_{\mathcal{M}} : \quad \begin{aligned} \phi_j^{(1)} &\mapsto -\phi_j^{(1)} - 2\phi_j^{(2)} - 4\tilde{\phi}_{j-1,j}^{(2)} + 8\tilde{\phi}_{j,j+1}^{(1)} \\ \phi_j^{(2)} &\mapsto -\phi_j^{(2)} + 4\tilde{\phi}_{j,j+1}^{(1)} \\ \tilde{\phi}_{j,j+1}^{(1)} &\mapsto -\tilde{\phi}_{j,j+1}^{(1)} \\ \tilde{\phi}_{j,j+1}^{(2)} &\mapsto -\tilde{\phi}_{j,j+1}^{(2)} + 2\tilde{\phi}_{j,j+1}^{(1)} \\ p_j^{(1)} &\mapsto -p_j^{(1)} \\ p_j^{(2)} &\mapsto -p_j^{(2)} + 2p_j^{(1)} \\ \tilde{p}_{j,j+1}^{(1)} &\mapsto -\tilde{p}_{j,j+1}^{(1)} - 2\tilde{p}_{j,j+1}^{(2)} - 4q_j^{(2)} + 8q_{j+1}^{(1)} \\ \tilde{p}_{j,j+1}^{(2)} &\mapsto -\tilde{p}_{j,j+1}^{(2)} + 4q_{j+1}^{(1)} \end{aligned}$$