Boundless Emanance in the Schwinger Model

Ross Dempsey

Based on:

- 2206.05308, with Igor Klebanov, Silviu Pufu, & Bernardo Zan
- 2508.16363, with Anna-Maria Glück, Silviu Pufu, & Benjamin Søgaard
- work in progress, with Shu-Heng Shao

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Axial symmetry and anomaly

Fermions in the continuum have vector and axial symmetries:

$$S = \int d^2x \, \left(i\overline{\psi}\gamma^\mu \partial_\mu \psi \right)$$

$$\mathrm{U}(1)_V : \quad \mathcal{U}^V(\alpha)\psi = e^{i\alpha}\psi, \qquad \mathrm{U}(1)_A : \quad \mathcal{U}^A(\alpha)\psi = e^{i\alpha\gamma^5}\psi.$$

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When we gauge $\mathrm{U}(1)_V$ in the continuum, $\mathrm{U}(1)_A$ is broken. In 1+1D:

$$\partial^\mu j^A_\mu = -\frac{1}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}.$$

This means $\mathcal{U}^A(\alpha)$ does not commute with the Hamiltonian.

Continuum Hamiltonian

After gauging, we have the Schwinger model (1+1D QED). Its Hamiltonian is

$$\mathcal{H}(\theta) = \int dx \left[\frac{g^2}{2} \left(E(x) + \frac{\theta}{2\pi} \right)^2 - i\psi^\dagger(x)\gamma^5(\partial_1 + iA_1(x))\psi(x) \right].$$

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Thus:

- Every $\mathcal{H}(\theta)$ is unitarily equivalent to $\mathcal{H}(0)$
- Fractional charges $\pm\frac{\theta}{2\pi}$ are screened by the dynamical fermions of charge ± 1

Goal: anomalous axial symmetry on the lattice

We would like a lattice analog of the anomalous axial transformation. It will come from translations, as was suggested long ago [Banks, Kogut, Susskind '75]:

> It is interesting that the axial charge $Q_5 = \int j_5^0(x)dx$ interchanges ψ_e with ψ_o. Therefore, it can be interpreted as a translation of the lattice by one unit in a. The momentum operator, on the other hand, has the continuum form, $p_1 = \int \psi(-i\partial_1)\psi dx$. Since the discrete form of the gradient ∂_1 connects only even or odd sites (cf. Eq. (2.4)), the discrete form of p_1 translates the lattice by 2a.

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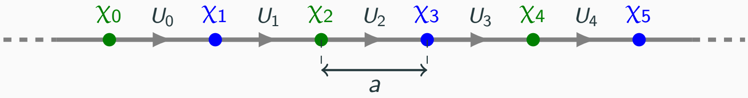
This talk: lattice models for which $TH(\theta)T^{-1} = H(\theta + 2\pi\frac{p}{r})$, giving $\mathbb{Z}_r$ for any order r

Staggered fermion Hamiltonian

[Kogut-Susskind '74]: stagger $\psi(x) = \begin{pmatrix} \psi_u(x) \\ \psi_d(x) \end{pmatrix}$ on alternating sites



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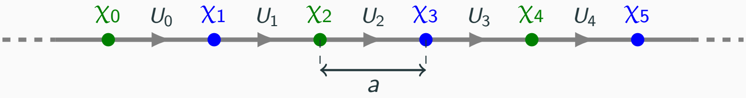

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We can discretize the continuum Hamiltonian:

$$\tilde{H}(\theta) = \frac{g^2 a}{2} \sum_n \left(L_n + \frac{\theta}{2\pi} \right)^2 - \frac{i}{2a} \sum_n \left(\chi_n^\dagger U_n \chi_{n+1} + \chi_n U_n^\dagger \chi_{n+1}^\dagger \right).$$

The gauge-invariance condition is typically taken to be

$$L_n - L_{n-1} = \chi_n^\dagger \chi_n + c_n, \qquad c_{2m} = 0, \qquad c_{2m+1} = -1;$$

note that c_n is a choice.

One-site translation

A one-site translation acting as $TL_n T^{-1} = L_{n+1}$ and $T\chi_n T^{-1} = \chi_{n+1}$ does not preserve the Gauss law. Instead,

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We then find

$$\begin{aligned} T\tilde{H}(\theta)T^{-1} &= \frac{g^2a}{2}\sum_n \left(L_n - c_n + \frac{\theta}{2\pi}\right)^2 + \text{hopping} \\ &= \frac{g^2a}{2}\sum_n \left(L_n + \frac{\theta+\pi}{2\pi} - \frac{(-1)^n}{2}\right)^2 + \text{hopping} \\ = \cdots &= \tilde{H}(\theta+\pi) - \frac{g^2a}{4}\sum_n (-1)^n\chi_n^\dagger\chi_n. \end{aligned}$$

Mass shift

We see that

$$H(\theta) \equiv \tilde{H}(\theta) - \frac{g^2 a}{8} \sum_n (-1)^n \chi_n^\dagger \chi_n$$

acts under translations the same way the continuum Hamiltonian acts under $\mathbb{Z}_2$ axial transformations [RD, Klebanov, Pufu, Zan '22]:

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- T^2 acts trivially in the IR $\to$ T gives emanant ([Cheng, Seiberg '22]), anomalous $\mathbb{Z}_2$ axial transformation

$\langle E \rangle(\theta)$ on the lattice

Continuum: anomaly enforces $\langle E \rangle(\theta) = \text{const}$, parity enforces $\langle E \rangle(-\theta) = -\langle E \rangle(\theta)$, so $\langle E \rangle(\theta) = 0$.

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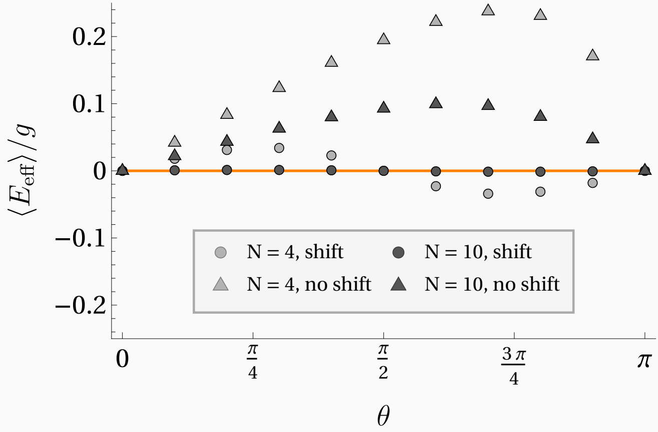
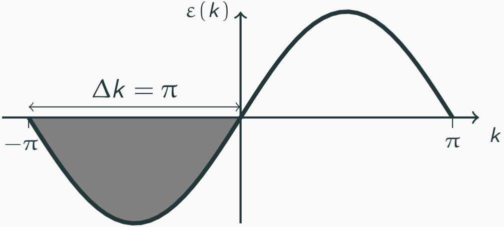


Translations and axial rotations

A free fermion Hamiltonian

$$H = -\frac{i}{2a}\sum_n \left(\chi_n^\dagger \chi_{n+1} + \chi_n \chi_{n+1}^\dagger\right) = a^{-1}\sum_k \sin k\, \tilde{\chi}_k^\dagger \tilde{\chi}_k$$

has a ground state with filling fraction $\nu = \frac{1}{2}$.



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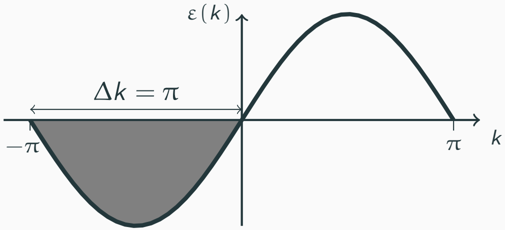

Translation acts as $e^{ik} \rightarrow$ right- and left-movers rotated by $\frac{\pi}{2}$ in opposite directions, up to an overall vector rotation. (See also [Chatterjee, Pace, Shao '24], Arko's talk)

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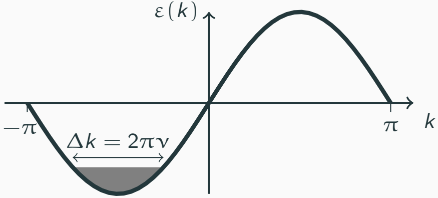
With a different filling fraction, translations emanate to axial rotations by $\pi\nu$:



Improved Hamiltonian

Following the derivation from before:

$$\sum_n \left(L_n - c_n + \frac{\theta}{2\pi} \right)^2 = \sum_n \left(L_n + \frac{\theta + 2\pi\nu}{2\pi} - (c_n + \nu) \right)^2 .$$

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Using that $c_n + \nu$ averages to zero, we can simplify this and find

$$T\tilde{H}(\theta)\,T^{-1} = \tilde{H}(\theta + 2\pi\nu) + g^2 a \sum_n \beta_n \, \chi_n^\dagger \chi_n ,$$

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We then find

$$H(\theta) = \frac{g^2 a}{2} \sum_n \left(L_n + \frac{\theta}{2\pi}\right)^2 - \frac{i}{2a\sin(\pi\nu)} \sum_n \left(\chi_n^\dagger U_n \chi_{n+1} + \chi_n U_n^\dagger \chi_{n+1}^\dagger\right)$$
$$+ g^2 a \sum_n \alpha_n\, \chi_n^\dagger \chi_n,$$

with α_n a cumulative sum of β_n, obeys

$$\boxed{TH(\theta)\,T^{-1} = H(\theta + 2\pi\nu)}.$$

Fractional charge screening

Let $\nu = \frac{p}{r}$, with e.g.

$$c_n = \begin{cases} -1 & 1 \leqslant n \leqslant p \pmod{r}, \\ 0 & p < n \leqslant r \pmod{r}. \end{cases}$$

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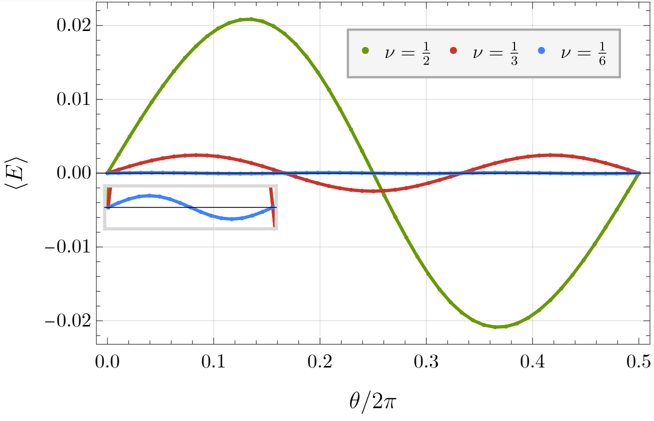
Equivalently: if fermions have charge q, we have $\mathbb{Z}_q^{[1]}$ distinguishing flux tube sectors and a $\mathbb{Z}_q$ remnant axial symmetry, and their mixed 't Hooft anomaly enforces screening of charge-1 probes (see [Honda, Itou, Tanizaki '22]). These symmetries and anomalies are realized exactly on the lattice when $q \mid r$.

$\langle E \rangle(\theta)$ again

On a 12-site periodic lattice, we can see the period of $\langle E \rangle(\theta)$ decrease as we increase the order r:



Infinite lattices

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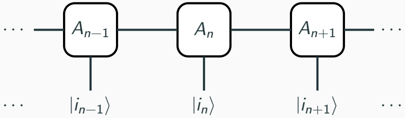
So, interlude: a new tensor network construction that's natural for gauge theories [RD, Glück, Pufu, Søgaard '25]

Uniform matrix product states

Matrix product state: contraction of three-index tensors to give a state in a tensor product Hilbert space



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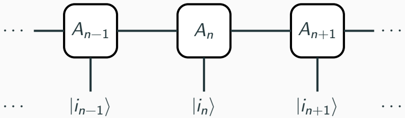
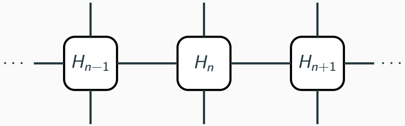


Matrix product operator: contraction of four-index tensors that acts on MPSs



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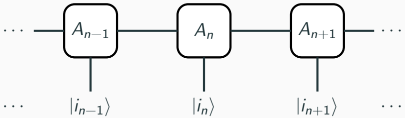

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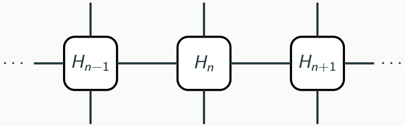
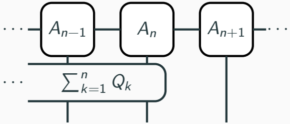


Uniform matrix product states/operators: restrict $A_n = A_{n+r}$ and $H_n = H_{n+r}$

Link-enhanced matrix product operators (LEMPOs)

For gauge theories: can solve for link operators L_n in terms of charge operators Q_n

$$L_n \, |A_1, \ldots, A_N\rangle =$$



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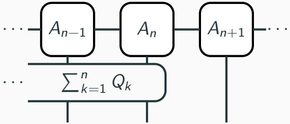

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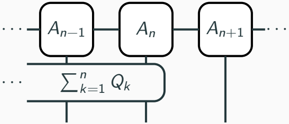

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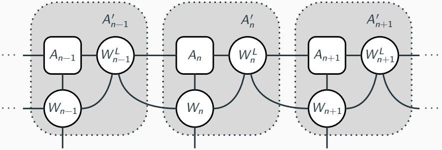
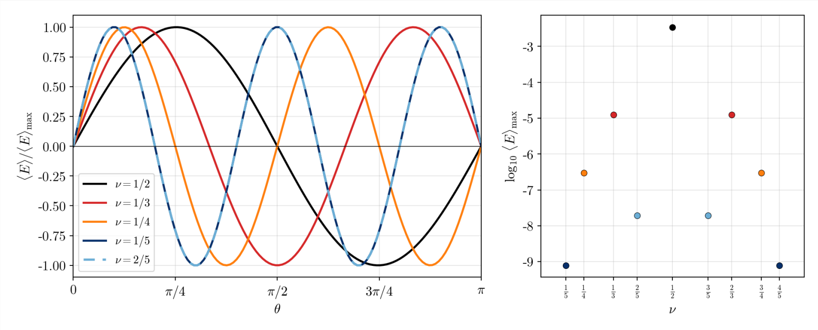
Link-enhanced matrix product operators: impose gauge-invariance condition on MPS tensors, and modify the MPO to act directly on links:

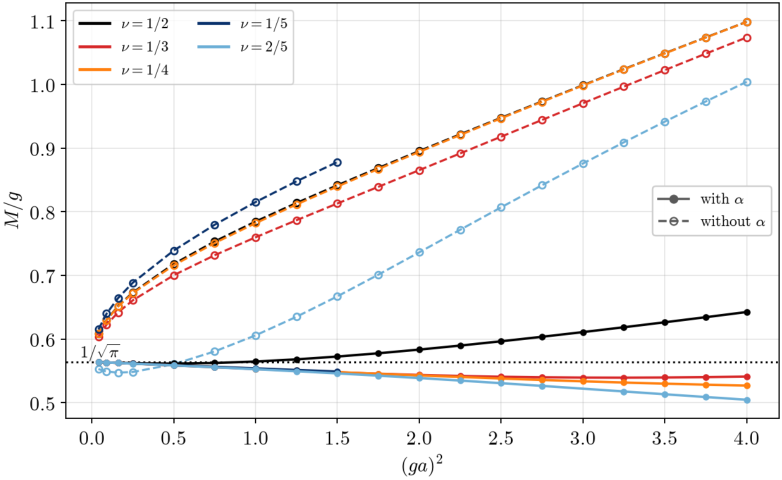


See also Benjamin's talk

$\langle E \rangle(\theta)$ one more time



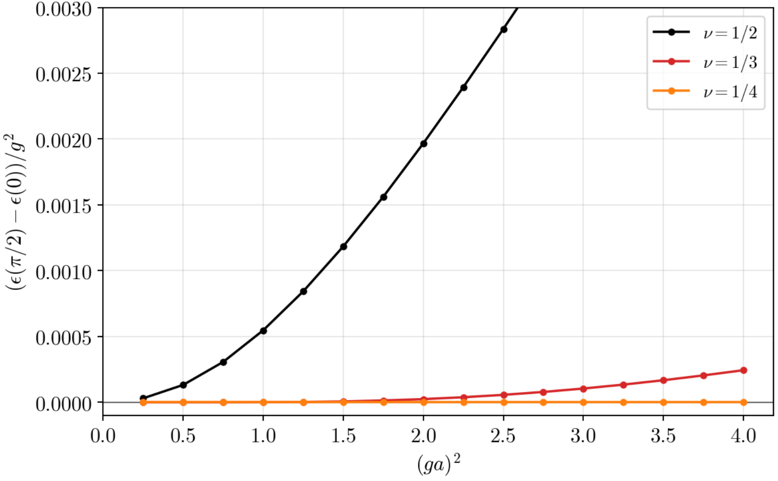
Mass gap



$\nu = 1/2$
$\nu = 1/5$
$\nu = 1/3$
$\nu = 2/5$
$\nu = 1/4$
with α
without α
$1/\sqrt{\pi}$
M/g
$(ga)^2$
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0

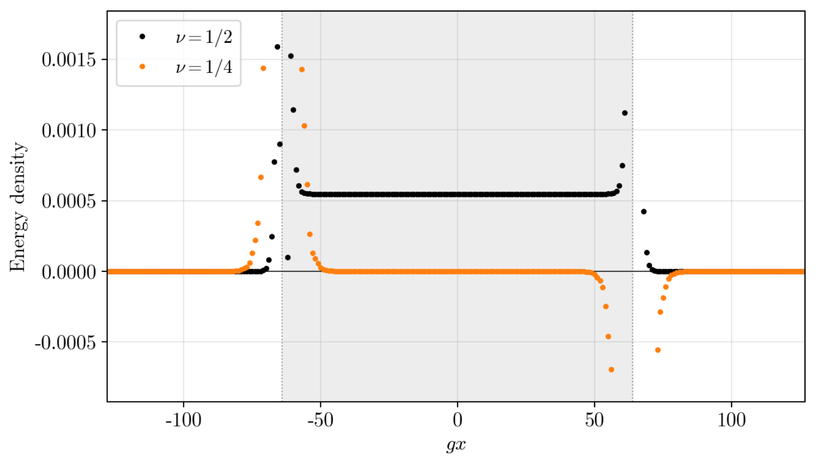
String tension

By comparing $\theta = 0$ with $\theta = \frac{\pi}{2}$, we see how the $\nu = \frac{1}{4}$ model screens a $\frac{1}{4}$ probe charge:



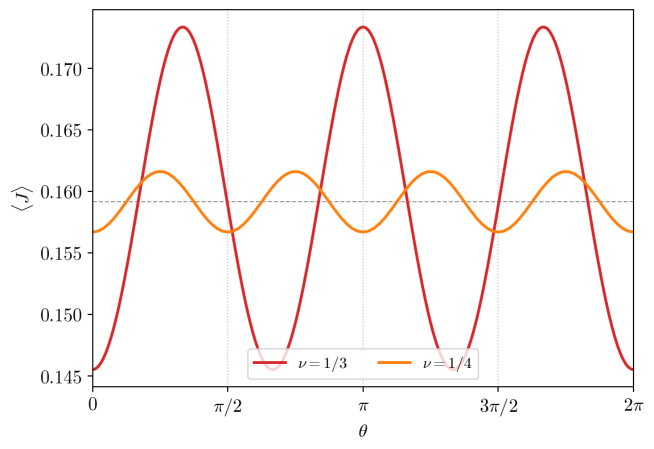
Probe charge screening

We can also see this by explicitly inserting charge-$\pm\frac{1}{4}$ defects on a large finite lattice:



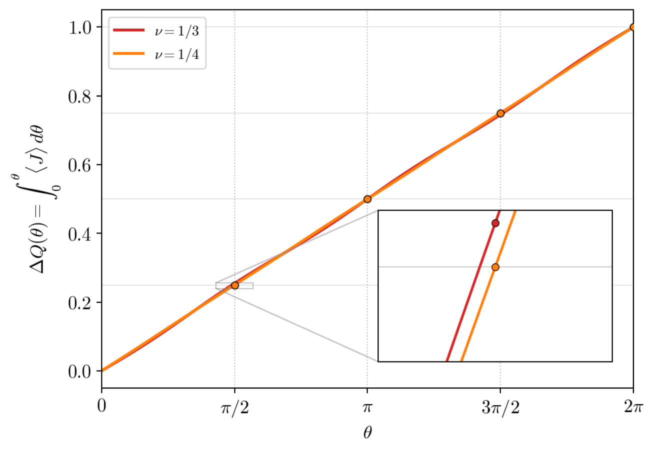
Charge pump

Every model pumps one unit of charge when $\theta \to \theta + 2\pi$; the $\nu = \frac{p}{r}$ models pump an exact fractional charge when $\theta \to \theta + \frac{2\pi}{r}$:



Charge pump

Every model pumps one unit of charge when $\theta \to \theta + 2\pi$; the $\nu = \frac{p}{r}$ models pump an exact fractional charge when $\theta \to \theta + \frac{2\pi}{r}$:



Outlook

- Axial symmetries are physically important even when broken by ABJ anomalies
- In the Schwinger model, the anomalous $\mathrm{U}(1)_A$ in the continuum can be realized as emanant $\mathbb{Z}_r$ transformations on the lattice, for arbitrary order r

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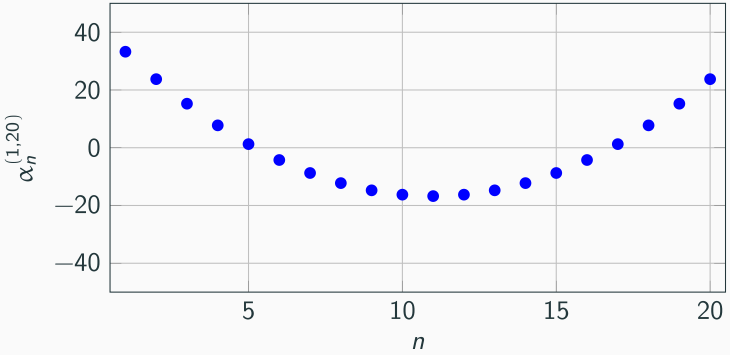
Next steps:

- Non-abelian 1+1D massless gauge theories also have vanishing string tensions. Can this be made exact on the lattice?

Adjustment coefficients

$$\alpha_n^{(p,r)} = \frac{1}{12r}\begin{cases} (r-p)(-6n^2+6(p+1)n+p(r-2p-3)) & 1 \leqslant n \leqslant p, \\ p(6n^2-6(r+p+1)n+(r+p)(r+2p+3)) & p < n \leqslant r \end{cases}$$



Mass term

$$H = H(m_{\text{im}} = 0) + \beta_{p/r}\,(-1)^{p-1}\left(m_{\text{im}} \sum_n e^{2\pi i(n-p/2)p/r}\chi_n^\dagger U_n \chi_{n+1} + \text{h.c.}\right)$$

where $\beta_{1/2} = \frac{1}{2}$ and $\beta_\nu = 1$ otherwise