Extended split property
or: How I learned to stop worrying and think of QFT as a lattice system

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New Developments in Lattice Fermions
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partially based on work in progress with C. Jones and R. Thorngren

Motivation

Question: Does a given quantum field theory admit a lattice realization?

Why do we care?

- **Practical:** we want to simulate a given QFT on a computer.
- **Conceptual:** QFT is not yet a well defined mathematical concept. Lattice systems provide a rigorous mathematical setting needed if one wants to prove some general statements.

Motivation

What do we mean by a lattice realization?

- The degrees of freedom live on a lattice with a microscopic scale ℓ (e.g. tensor product of qubits over $\mathbb{Z}^d$).
- The interactions between degrees of freedom are assumed to have finite-range (or rapid decay) that bounds degrees of freedom together and produces an interesting long-distance behavior.
- Under favorable circumstances, a field theoretic description appears when a proper scaling limit $\ell \to 0$ is taken (with an appropriate scaling of coupling constants).

Motivation

At the level of kinematics, there should be an identification of observables and their symmetries in QFT and on the lattice.

It is, however, hard to compare observables in general, as the QFT description usually appears only in the limit $\ell \to 0$ when lattice disappears.

In this talk, we address a sharper question:

Question: When does the algebra of observables of a given quantum field theory admit a tensor product factorization? In other words, when does a QFT admit an exact lattice realization?

Algebra of observables in quantum mechanics

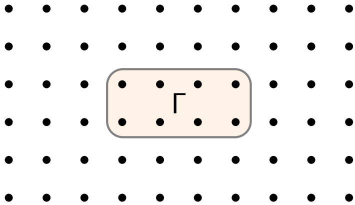
In quantum mechanics, the algebra of observables is defined by the algebra of bounded operators $B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, or a Hilbert superspace if fermionic degrees of freedom are present.

An operator $a : \mathcal{H} \to \mathcal{H}$ is bounded $a \in B(\mathcal{H})$ if it is a continuous linear map defined on the whole Hilbert space.

Unbounded operators are also physically relevant (e.g. coordinate $\hat{x}$ and momentum $\hat{p}$ operators), but they do not form an algebra. We can deal with them by considering bounded functions of such operators (e.g. $e^{i\kappa\hat{x}}$, $e^{i\xi\hat{p}}$ for $\kappa, \xi \in \mathbb{R}$).

Algebra of observables of a lattice system

A tensor product lattice system is defined by assigning infinite-dimensional Hilbert spaces $\{\mathcal{V}_x\}_{x\in\mathbb{Z}^d}$ to sites of a lattice $\mathbb{Z}^d \subset \mathbb{R}^d$.



The algebra of observables $\mathcal{A}$ is formally an infinite tensor product

$$\mathcal{A} = \bigotimes_{x\in\mathbb{Z}^d} B(\mathcal{V}_x).$$

More precisely, it is the algebra generated by $B\left(\otimes_{x\in\Gamma}\mathcal{V}_x\right)$ for finite subsets $\Gamma \subset \mathbb{Z}^d$. For fermionic systems, we take the graded tensor product of Hilbert superspaces.

Algebra of observables of a QFT: Minkowski space

We use the standard algebraic quantum field theory approach.

Let $\mathcal{H}$ be the Hilbert space of the vacuum state $|\Omega\rangle \in \mathcal{H}$.

The algebra of observables localized in a bounded region $O \subset \mathbb{R}^{d,1}$, e.g. smeared fermionic field insertions

$$\psi_f = \int dx\, \psi(x) f(x), \quad \mathrm{supp}(f) \subset O,$$

forms a subalgebra $\mathcal{A}_O \subset B(\mathcal{H})$. In fact, it is a von Neumann algebra that can be characterized by the following relations

$$\mathcal{A}'_O = \mathrm{Comm}(\mathcal{A}_O, B(\mathcal{H})), \;\; \mathcal{A}_O = \mathrm{Comm}(\mathcal{A}'_O, B(\mathcal{H})).$$

Physically, it means that if an infinite sequence of observables $a_1, a_2, \ldots \in \mathcal{A}_O$ has convergent matrix elements, then it has a limit inside $\mathcal{A}_O$, as was shown by von Neumann.

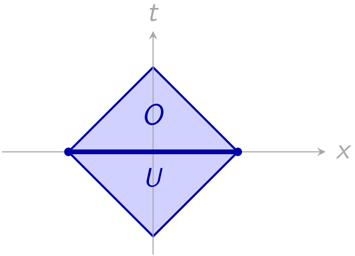
Algebra of observables of a QFT: Minkowski space

These von Neumann algebras $\mathcal{A}_O \subset B(\mathcal{H})$ satisfy the following natural properties:

- **Isotony:** $O_1 \subset O_2$ implies $\mathcal{A}_{O_1} \subset \mathcal{A}_{O_2}$.
- **Causality:** for space-like separated $O_1, O_2 \subset \mathbb{R}^{d,1}$, we have $[\mathcal{A}_{O_1}, \mathcal{A}_{O_2}]_\pm = 0$.
- **Poincaré symmetry:** a representation of the Poincaré group P that maps $\mathcal{A}_O$ to $\mathcal{A}_{O^g}$ for any $g \in P$.

Algebra of observables of a QFT: time-slice

In the Hamiltonian picture, we associate subalgebras with bounded regions of a constant time-slice $\mathbb{R}^d$.



The subalgebras $\mathcal{A}_U$ satisfy the following natural properties:

- **Isotony:** $U_1 \subset U_2$ implies $\mathcal{A}_{U_1} \subset \mathcal{A}_{U_2}$.
- **Locality:** for disjoint U_1, U_2, we have $[\mathcal{A}_{U_1}, \mathcal{A}_{U_2}]_{\pm} = 0$.
- **Poincaré symmetry:** a representation of the Poincaré group P that maps $\mathcal{A}_U$ to $\mathcal{A}_{U^g}$ for any Euclidean symmetry $g \in E \subset P$.

Example: $(1+1)D$ free Majorana fermion

The global algebra of observables is generated by $\chi_f^{(a)}$, $a = 1, 2, ..., N$, which are smeared fermionic operators

$$\chi_f^{(a)} = \int_{\mathbb{R}} \chi^{(a)}(x) f(x) dx$$

for a square-integrable test function $f \in L^2(\mathbb{R})$.

These generators satisfy the following anti-commutation relations

$$\{\chi_f^{(a)}, \chi_g^{(b)}\} = \delta^{ab} \int_{\mathbb{R}} dx\, f(x) g(x).$$

Note that in contrast to point-like insertions $\chi^{(a)}(x)$, the smeared observables $\chi_f^{(a)}$ are bounded.

Example: $(1+1)D$ free Majorana fermion

Let's consider a single massless Majorana fermion, for which $a = 1, 2$ (left and right moving modes).

Let $\mathcal{H}$ be the vacuum Hilbert space with the vacuum vector $|\Omega\rangle \in \mathcal{H}$ defining the ground state. For the two-point function, we have

$$\langle \Omega | \chi^{(a)}(x) \chi^{(b)}(y) | \Omega \rangle = I^{ab} \frac{1}{2} \delta(x-y) + Z^{ab} \frac{1}{2\pi i} \text{p.v.} \frac{1}{x-y}$$

The algebra $\mathcal{A}_I$ associated with an interval $I \subset \mathbb{R}$ is the von Neumann algebra generated by $\chi_f^{(a)}$ acting on $\mathcal{H}$ for f supported inside I.

Remark: Note that the von Neumann algebras $\mathcal{A}_I$ depend on the ground state and the associated representation. The algebra generated by $\chi_f^{(a)}$ abstractly has different representations, and a different ground state may give rise to a representation that is not unitarily equivalent.

A unifying framework: quasi-local algebra

An unifying framework for both the lattice and the continuum is the framework of quasi-local algebras, originally introduced by Haag and Kastler for algebraic QFT.

Let X be a metric space. A quasi-local algebra over X is a C^*-algebra $\mathcal{A}$ with von Neumann subalgebras $\mathcal{A}_U \subset \mathcal{A}$ associated with bounded subsets $U \subset X$, satisfying

1. If $U \subset V$, then $\mathcal{A}_U \subset \mathcal{A}_V$;
2. If $U \cap V = \emptyset$, then $[\mathcal{A}_U, \mathcal{A}_V]_\pm = 0$.

Lattice systems correspond to $X = \mathbb{Z}^d$. Continuum systems correspond to $X = \mathbb{R}^d$, possibly with an extra structure such as Euclidean or Poincaré symmetry.

Lattice vs. QFT

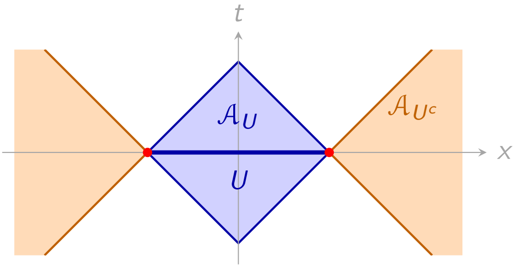
Let $\mathcal{A}_{QFT}$ be a quasi-local algebra of a $(d+1)$-dimensional QFT over $\mathbb{R}^d$, and $\mathcal{A}_{lattice} = \bigotimes_{x \in \mathbb{Z}^d} B(\mathcal{V}_x)$ be a quasi-local algebra of a lattice system over $\mathbb{Z}^d$.

Question: Is there an isomorphism $\varphi : \mathcal{A}_{QFT} \to \mathcal{A}_{lattice}$ that preserves locality?

What sort of locality do we want to preserve?
Options: strict locality, exponential locality, ...

Absence of factorization in QFT

It is well-known that there is infinite entanglement between any two neighboring regions in quantum field theory due to UV divergences.



It leads to the following no-go result:

Theorem (Araki '64, Fredenhagen '85)

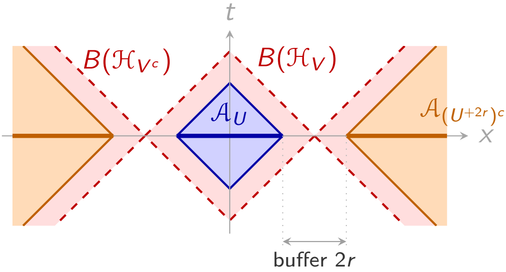
For any bounded $U \subset \mathbb{R}^d$*, there is no tensor product factorization of a Hilbert space* $\mathcal{H} = \mathcal{H}_U \otimes \mathcal{H}_{U^c}$ *such that* $\mathcal{A}_U \subset B(\mathcal{H}_U)$ *and* $\mathcal{A}_{U^c} \subset B(\mathcal{H}_{U^c})$.

Split property

However, a tensor product factorization exists if one allows some buffer region between U and U^c. More precisely,

Theorem (Buchholz, Wichmann '86)

For any bounded $U \subset \mathbb{R}^d$ and $V = U^{+r}$, there exists a tensor product factorization of a Hilbert space $\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_{V^c}$ such that $\mathcal{A}_U \subset B(\mathcal{H}_V)$ and $\mathcal{A}_{(U^{+2r})^c} \subset B(\mathcal{H}_{V^c})$.



It has been proven for any quantum field theory satisfying normal thermodynamic properties (no Hagedorn type phase transitions).

Extended split property

We say that a map $\mathcal{A} \to \mathcal{B}$ between quasi-local algebras has bounded spread if there exists $r > 0$, such that for any bounded $U \subset \mathbb{R}^d$, the image of $\mathcal{A}_U$ is contained in $\mathcal{B}_{U^{+r}}$.

Let $\mathcal{A}_{QFT}$ be a quasi-local algebra of a QFT, and $\mathcal{A}_{lattice} = \bigotimes_{x \in \mathbb{Z}^d} B(\mathcal{V}_x)$ be a quasi-local algebra of a lattice system.

Question: Is there a bounded spread isomorphism

$$\varphi_{\text{bs}} : \mathcal{A}_{QFT} \to \mathcal{A}_{lattice}?$$

More generally, one can ask if there is an isomorphism

$$\varphi_{\text{exp}} : \mathcal{A}_{QFT} \to \mathcal{A}_{lattice},$$

such that $\varphi_{\text{exp}}^{-1}(B(\mathcal{V}_x)) \subset \mathcal{A}_{QFT}$ is exponentially localized near $x \in \mathbb{R}^d$?

$(1+1)D$ Majorana fermion

Recall that the quasi-local algebra is generated by $\chi_f^{(1)}, \chi_f^{(2)}$ acting on a Hilbert space $\mathcal{H}$ with the vacuum vector $|\Omega\rangle \in \mathcal{H}$ satisfying

$$\langle\Omega|\chi^{(a)}(x)\chi^{(b)}(y)|\Omega\rangle = I^{ab}\,\frac{1}{2}\delta(x-y) + Z^{ab}\,\frac{1}{2\pi i}\mathrm{p.v.}\frac{1}{x-y}.$$

We can represent the algebra on a different Hilbert space $\mathcal{H}_0$ with a vector $|\Psi\rangle \in \mathcal{H}_0$ such that

$$\langle\Psi|\chi^{(a)}(x)\chi^{(b)}(y)|\Psi\rangle = \frac{I^{ab} + Y^{ab}}{2}\,\delta(x-y).$$

One can think of this state as a ground state of an infinitely massive fermion. Due to absence of correlations, the resulting quasi-local algebra factorizes $\mathcal{A}^{(0)} \cong \bigotimes_{x\in\mathbb{Z}} B(\mathcal{V}_x)$, where $B(\mathcal{V}_x)$ is generated by modes supported in $[x-1/2, x+1/2]$.

$(1+1)D$ Majorana fermion

In the momentum space, two-point functions have the form

$$G_\Omega(k) = \frac{I + Z\,\mathrm{sign}(k)}{2}$$

$$G_\Psi(k) = \frac{I + Y}{2}$$

By applying a linear change of variables $\chi(k) \to V(k)\chi(k)$ for a unitary

$$V(k) = \frac{1}{\sqrt{2}}(1 + iX\,\mathrm{sign}(k)) + \mathcal{O}(|k|^{-\infty})$$

we get

$$V(k) \cdot G_\Omega(k) \cdot V(k)^\dagger = G_\Psi(k) + \mathcal{O}(|k|^{-\infty}).$$

which has the same behavior at short distances as the state $|\Psi\rangle$.

$(1+1)D$ Majorana fermion

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$$G_\Omega(k) = \frac{I + Z\,\mathrm{sign}(k)}{2}$$

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By applying a linear change of variables $\chi(k) \to V(k)\chi(k)$ for a unitary

$$\boxed{V(k) \;=\; e^{\,i\frac{\pi}{4}\hat{n}(k)\cdot\vec{\sigma}} \;=\; \frac{1}{\sqrt{2}}\Big(1 + i\tanh(k/\mu)\,X + i\,(\cosh(k/\mu))^{-1}\,Y\Big)}$$

we get

$$V(k)\cdot G_\Omega(k)\cdot V(k)^\dagger = G_\Psi(k) + \mathcal{O}(e^{-|k|/\mu}).$$

which has the same behavior at short distances as the state $|\Psi\rangle$.

$(1+1)D$ Majorana fermion

In the coordinate space, the observables are mapped to

$$\chi^{(a)}(x) \to \int_{\mathbb{R}} dy\, \tilde{V}^{(ab)}(x-y)\chi^{(b)}(y)$$

which are exponentially localized near the point x.

The resulting map induces an isomorphism preserving exponential locality

$$\varphi_{\mathrm{exp}} : \mathcal{A}_{\mathrm{Maj}} \to \bigotimes_{x\in\mathbb{Z}} B(\mathcal{V}_x).$$

Each factor $B(\mathcal{V}_x) \subset \mathcal{A}_{\mathrm{Maj}}$ is exponentially localized near x.

$(1+1)D$ Majorana fermion

In fact, for any $\varepsilon > 0$, one can choose

$$V(k) = \frac{1}{\sqrt{2}}(1 + iX\,\mathrm{sign}(k)) + \mathcal{O}(|k|^{-\infty})$$

such that $\tilde{V}(x)$ is supported in $[-\varepsilon, \varepsilon]$. Then

$$\chi^{(a)}(x) \to \int_{\mathbb{R}} dy\, \tilde{V}^{(ab)}(x-y)\chi^{(b)}(y)$$

provides a bounded spread isomorphism

$$\varphi_{\mathrm{bs}} : \mathcal{A}_{\mathrm{Maj}} \to \bigotimes_{x\in\mathbb{Z}} B(\mathcal{V}_x).$$

But an explicit expression for $V(k)$ is more complicated than in the case of exponential localization.

$(1+1)D$ chiral Majorana fermion: no-go theorem

For chiral fermions, the construction above doesn't work. The following no-go theorem can be proven for free fermions with non-trivial gravitational anomaly c_-:

Theorem

Let $\mathcal{A}$ be the quasi-local algebra of a $(1+1)D$ Majorana-Weyl fermion. Then there is no bounded spread isomorphism $\mathcal{A} \to \bigotimes_{x\in\mathbb{Z}} B(\mathcal{V}_x)$.

Note that, in contrast to Nielsen-Ninomiya type theorems, this result is non-perturbative and excludes the possibility of any non-linear realization.

It is true, however, that one can always tensor the algebra of a left-moving Majorana-Weyl fermion with another quasi-local algebra (of a right-moving fermion) such that the resulting algebra factorizes.

Invertible quasi-local algebras

That motivates us to introduce the following

Definition

We say that a quasi-local algebra $\mathcal{A}$ is invertible if there exists another quasi-local algebra $\mathcal{B}$, such that $\mathcal{A} \otimes \mathcal{B}$ is bounded spread isomorphic to a tensor product algebra $\bigotimes_{x \in \mathbb{Z}} B(\mathcal{V}_x)$.

Such quasi-local algebras have been introduced in [Haah '22] in the context of lattice systems with finite-dimensional Hilbert spaces as a concept relevant for studying quantum cellular automata.

Invertible quasi-local algebras

Conjecture

The quasi-local algebra of any genuine quantum field theory is invertible. It is bounded spread isomorphic to a tensor product algebra if and only if the gravitational anomalies vanish.

This conjecture is reminiscent of [Cordova, Freed, Lam, Seiberg '19].

Evidence:

- Tensor product factorization for anomaly-free free fermionic theories.
- Tensor product factorization for any rational diagonal CFT.
- Invertibility of fusion spin chain algebras [Bunner, Jones '26]
- Consistent with an expectation that a genuine QFT does not need any attached theory in the bulk.

Application: anomalous symmetries

(thanks to conversations with Sahand Seifnashri)

For many anomalous symmetries of QFT, there are issues with realizing them on a lattice system with finite-dimensional Hilbert spaces. This includes $\mathbb{Z}_2$-symmetry for $(1+1)D$ fermions, Lie group symmetries, etc.

Our construction provides a way to realize such symmetries on a lattice system with infinite-dimensional Hilbert spaces via tensor product factorization of $\mathcal{A}_{QFT}$.

We illustrate it on an example of $SO(N)$ symmetry of a $(1+1)D$ massless Majorana fermion transforming in the fundamental representation. To our knowledge, this is the first lattice realization of an anomalous non-abelian continuous symmetry.

Application: anomalous symmetries

Let $\mathcal{A}$ be the quasi-local algebra of $(1+1)D$ massless Majorana fermion transforming in the fundamental representation of $SO(N)$. There is a locality preserving isomorphism

$$\varphi : \mathcal{A} \to \bigotimes_{x \in \mathbb{Z}} B(\mathcal{V}_x)$$

and an action of symmetry $\gamma^{(g)} : \mathcal{A} \to \mathcal{A}$ on left-movers only for $g \in SO(N)$. Then

$$\tilde{\gamma}^{(g)} := \varphi \circ \gamma^{(g)} \circ \varphi^{-1}$$

defines an action $\tilde{\gamma}^{(g)} : \bigotimes_{x \in \mathbb{Z}} B(\mathcal{V}_x) \to \bigotimes_{x \in \mathbb{Z}} B(\mathcal{V}_x)$ which is locality preserving and has the same anomaly as in field theory.

Application: anomalous $SO(3)$ symmetry on the lattice

Take three Majorana fermions $\chi^{(a)}$ with $SO(3)$ rotating them, acting on the left-movers only:

$$j^a(x) = \frac{i}{2}\,\epsilon^{abc}\,\chi_L^{(b)}(x)\,\chi_L^{(c)}(x), \qquad \hat{Q}^a_f = \int dx\, f(x)\, j^a(x).$$

These generate $\mathfrak{so}(3)_1 \cong \mathfrak{su}(2)_2$ at level one:

$$[\hat{Q}^a_f, \hat{Q}^b_g] = i\,\epsilon^{abc}\,\hat{Q}^c_{fg} \;+\; \frac{\delta^{ab}}{2\pi i}\int dx\, f'(x)\, g(x).$$

Each $B(\mathcal{V}_x)$, $x \in \mathbb{Z}$ is generated by Majorana modes $\{\eta_{x,j}\}_{j\in\mathbb{N}}$ with $\{\eta_{x,j}, \eta_{y,k}\} = \delta_{xy}\delta_{jk}$. Applying φ,

$$\tilde{Q}^a_f := \varphi(\hat{Q}^a_f) = \frac{i}{4}\sum_{x,y\in\mathbb{Z}}\;\sum_{j,k\in\mathbb{N}} \Lambda^{a,f}_{xj,yk}\;\eta_{x,j}\,\eta_{y,k},$$

with $\Lambda^{a,f}$ real antisymmetric, $|\Lambda^{a,f}_{xj,yk}| \le C\,e^{-|x-y|/\xi}$. The central term obstructs an on-site action: no φ makes g act as $\prod_n u_n(g)$.

Remarks

- Our setting also provides a way to realize anomalous symmetries of higher dimensional free fermionic theories.
- In particular, it provides a way to realize a mixed gravitational anomaly in 3+1 dimensions.
- Infinite dimensionality of the on-site Hilbert spaces is an essential ingredient of the construction.

Summary

- We have discussed whether there exists a locality preserving isomorphism $\mathcal{A}_{QFT} \to \mathcal{A}_{lattice}$.
- We provided an explicit construction for a $(1+1)D$ free Majorana fermion.
- The gravitational anomaly c_- of $(1+1)D$ theories is an obstruction for such a factorization.
- We conjectured that for all genuine QFTs, the quasi-local algebra is invertible, and that extended split property holds if and only if the gravitational anomaly vanishes.
- We have shown how tensor factorization of $\mathcal{A}_{QFT}$ can be used to provide a lattice realization of anomalous symmetries of QFTs. In particular, it provides a realization of anomalous non-abelian Lie group symmetries.