Bosonization versus the Nielsen-Ninomiya theorem

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New Developments in Lattice Fermions
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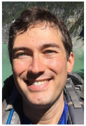

Aleksey Cherman

A saying

A raisin at the end of the sausage

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A raisin at the end of the sausage

Introduction

As we saw earlier this week [see Sahand's talk], bosonization is one tool to put chiral fermions on the lattice

Question we want to answer:

Can a lattice bosonic model reproduce chiral fermions with no doublers?

In particular, we will consider the Lagrangian formulation of the **modified Villain model**

Nielsen-Ninomiya (NN) Theorem

NN theorem: any discretization of chiral fermions with "nice peroperties" leads to doublers

- Originally formulated for free fermions in a Hamiltonian system
- There is a family of NN-type theorems [Fidkowski & Xu 2023, Shamir 1993, ...]

How exactly does it apply to the modified Villain scalar model?

- (Euclidean) Lagrangian description, could be interacting, infinite DOF per site, ...

Nielsen-Ninomiya (NN) Theorem

With some tweaks, we formulate a statement of an Euclidean NN-type theorem with **no reference** to

- fermionic fields as the fields of the path integral
- the number of DOF per site
- the interactions of the (would-be) fermions

Then, we study its consequences in the **modified Villain model**

Our statement of the Nielsen-Ninomiya Theorem

Consider a lattice Dirac kernel $\not{D}(x,y)$ with $x, y \in \mathbb{Z}^n$ satisfying:

(A) **Translation-invariance:** $\not{D}(x,y)$ is invariant under lattice translation

(B) **Chirality:** $\not{D}(x,y) = \gamma^\mu D_\mu(x,y)$ for some hermitian $D_\mu(x,y)$

(C) **Quasi-locality:** $\not{D}(x,y)$ has integrable decay in $|x-y|$

(D) **Desired chiral fermion:** the Fourier transform $\widetilde{\not{D}}(p)$ of $\not{D}(x,y)$ has an isolated singularity at $p=0$ whose phase winds by 1

Then, we find a doubler(s):
$\widetilde{D}(p)$ has **at least one other zero** somewhere with $p \neq 0$, such that the sum of indices of all zeros equals 0

Proof follows e.g. [Karsten 1981]

Review of Coleman's 2d boson-fermion duality

Interpretation to keep in mind: a boson-fermion duality relates two path-integral descriptions of the same theory, one written with bosonic fields and the other with fermionic ones

2d free massless Dirac fermion:

$$\int \mathcal{D}\psi_\pm \mathcal{D}\bar{\psi}_\pm \exp\left\{-\frac{1}{\pi}\int \mathrm{d}^2x\left(\bar{\psi}_+\bar{\partial}\psi_+ + \bar{\psi}_-\partial\psi_-\right)\right\}$$

- $U(1)_L \times U(1)_R \equiv \frac{U(1)_V \times U(1)_A}{\mathbb{Z}_2}$ has 't Hooft anomaly
- For the duality map to the bosonic theory, we take fermion number $(-1)^F$ to be gauged

Review of Coleman's 2d boson-fermion duality

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2d compact scalar:

$$\int \mathcal{D}\phi \, \exp\left\{-\frac{1}{8\pi}\int \mathrm{d}^2x \, |\partial_\mu \phi|^2\right\}$$

- Shift and winding symmetry $U(1)_S \times U(1)_W$ has 't Hooft anomaly
- θ is the T-dual of ϕ

Duality dictionary

$$\psi_+ \Longleftrightarrow e^{i\theta+i\phi/2}$$

Note it is a non-local operator on both sides of the duality

- ψ_+ charged under fermion number
- $e^{i\phi/2}$ charged under $\phi \to \phi + 2\pi$

Attaching to a $\mathbb{Z}_2$ topological line restores gauge invariance

We want to use the bosonic description to calculate fermionic quantities on the lattice

Modified Villain model

$$\mathcal{S}(\varphi, n, \theta) = \frac{1}{8\pi} \sum_{\text{links}\,\ell} \Big[\mathrm{d}\varphi(\ell) + 2\pi n(\ell) \Big]^2 + i \sum_{\text{plaquettes}\,p} \theta(p) \mathrm{d}n(p)$$

- φ is an $\mathbb{R}$ scalar made compact by gauging 2π shifts,

$$\varphi \to \varphi + 2\pi k, \quad n \to n - \mathrm{d}k \qquad n, k \in \mathbb{Z}$$

- $\theta = \theta + 2\pi$ is the T-dual counterpart to φ
  - φ lives on sites, θ on plaquettes
- the $\sum_p$ term acts to suppress winding vortices $\mathrm{d}n$, i.e. making $U(1)_W$ a good symmetry

What question are we trying to answer?

Via the bosonization map and the modified Villain formalism, can write down fermionic quantities on the lattice

Study the Dirac operator in this language:

How is it consistent with NN theorem?

To get the Dirac operator, need to calculate the Weyl 2-pt function

$$D^{-1} \sim \left\langle \psi_+(x)\bar{\psi}_+(y) \right\rangle$$

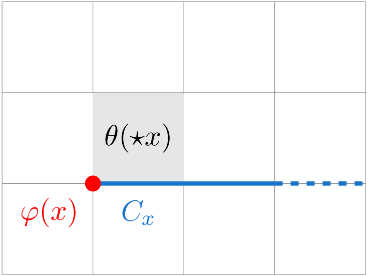
in terms of the bosonic variables

Weyl operators in the modified Villain model

Recall $\psi_+ \sim e^{i\theta + i\phi/2}$ needs to be attached to a $\mathbb{Z}_2$ topological line for gauge invariance

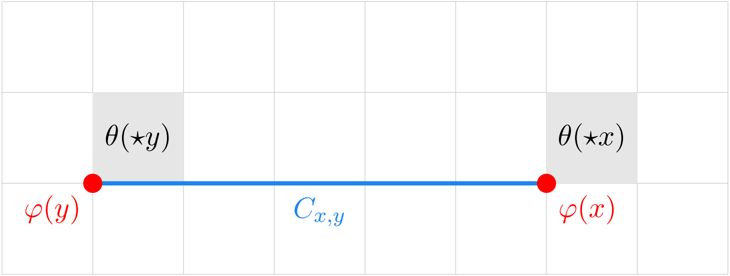
On the lattice:

$$\psi_+(x) \sim e^{i\theta(\star x) + \frac{i}{2}\varphi(x)} \exp\left[i\pi \sum_\ell C_x(\ell) n(\ell)\right]$$



Weyl 2-point function

$$\psi_+(x)\bar{\psi}_+(y) \sim e^{i\varphi(x)/2+i\theta(\star x)-i\varphi(y)/2-i\theta(\star y)} \exp\left[i\pi \sum_\ell C_{x,y}(\ell)n(\ell)\right]$$



Weyl 2-point function, position space

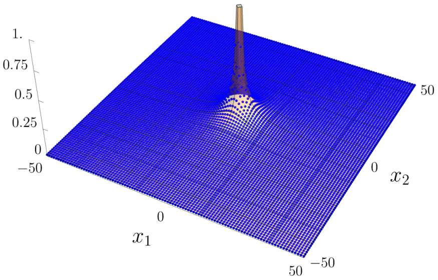
Result:

$$\left\langle \psi_+(x)\bar{\psi}_+(y)\right\rangle \sim \exp\left\{2\pi G(x-y)+i\pi\sum_z \mathrm{d}C_{x,y}(\star z)\left[G\left(x-z\right)-G\left(y-z\right)\right]\right\}$$

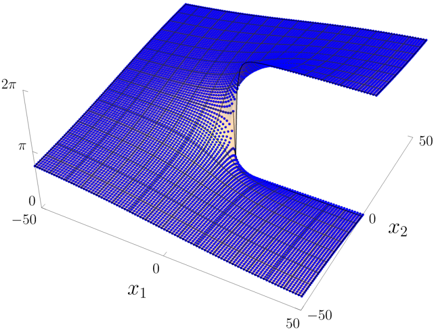
Weyl 2-point function, position space

Result:

$$\left\langle \psi_+(x)\bar{\psi}_+(y)\right\rangle \sim \exp\left\{2\pi G(x-y)+i\pi\sum \mathrm{d}C_{x,y}(\star z)\left[G\left(x-z\right)-G\left(y-z\right)\right]\right\}$$

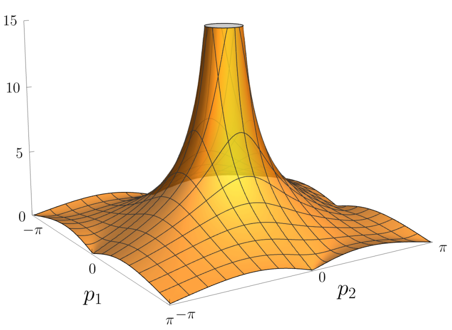


(a) $\left|\left\langle \psi_+(x_1,x_2)\bar{\psi}_+(0,0)\right\rangle\right|$

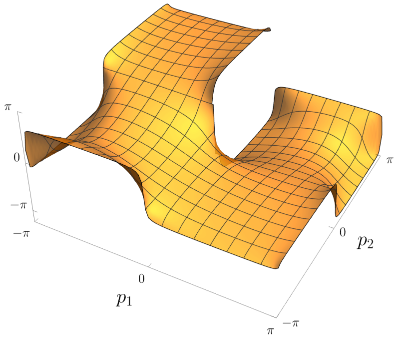


(b) $\arg\left\langle \psi_+(x_1,x_2)\bar{\psi}_+(0,0)\right\rangle$

Weyl 2-point function, momentum space



(a) $\left|\widetilde{\langle\psi_+\bar\psi_+\rangle}(p_1,p_2)\right|$

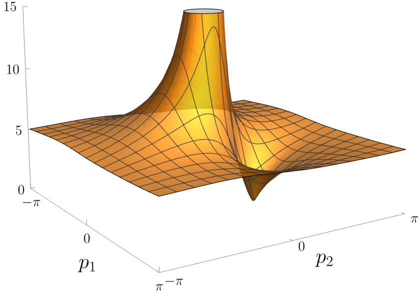


(b) $\arg\widetilde{\langle\psi_+\bar\psi_+\rangle}(p_1,p_2)$

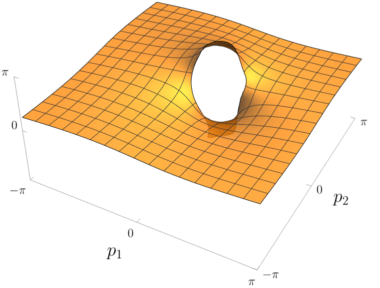
- Pole at $(0,0)$ has index -1
- Zeros at $(\pi,0),(0,\pi),(\pi,\pi)$ have indices $+1,+1,-1$, respectively

Note: exact number of zeros is not fixed by theorem

- Just require indices to cancel
- Changing position-space contact term changes position and number of zeros, e.g.:



(a) $\left|\langle\widetilde{\psi_+\bar{\psi}_+}\rangle(p_1,p_2)\right|$



(b) $\arg\langle\widetilde{\psi_+\bar{\psi}_+}\rangle(p_1,p_2)$

Interpretation in terms of the Dirac operator

$$\langle \psi_+(x)\bar{\psi}_+(y)\rangle \sim D^{-1}$$

No extra poles in the 2-point function $\implies$ **no doublers**

Zeros in the 2-point function $\implies$ **nonlocality** in the Dirac operator

- Note the microscopic description is still manifestly local

As expected, **consistent with the NN theorem**

Another observation: the fermions are interacting

Consider connected 4-pt function, which would vanish in the free theory

- E.g. with four fermions on the corners of a large box of size L,

$$\left|\left\langle\psi_\pm(C_x)\psi_\pm(C_z)\bar{\psi}_\pm(C_y)\bar{\psi}_\pm(C_w)\right\rangle\right| \sim \frac{1}{L^2} - \frac{5}{24L^4} + \mathcal{O}\left(\frac{1}{L^6}\right)$$

$$\left|\left\langle\psi_\pm(C_x)\bar{\psi}_\pm(C_w)\right\rangle\left\langle\psi_\pm(C_z)\bar{\psi}_\pm(C_y)\right\rangle - \left\langle\psi_\pm(C_x)\bar{\psi}_\pm(C_y)\right\rangle\left\langle\psi_\pm(C_z)\bar{\psi}_\pm(C_w)\right\rangle\right| \sim \frac{1}{L^2} - \frac{1}{12L^4} + \mathcal{O}\left(\frac{1}{L^6}\right)$$

Therefore $\left\langle\psi_\pm(C_x)\psi_\pm(C_z)\bar{\psi}_\pm(C_y)\bar{\psi}_\pm(C_w)\right\rangle_{\text{connected}} \neq 0$

$\implies$ Bosonic model is Gaussian, but the fermions are interacting
$\implies$ Mismatch vanished in the continuum limit; recover free fermions

Summary/outlook

NN forbids the existence of a local, translation-invariant, doubler-free chiral Dirac kernel

- Doesn't forbid a local lattice model from having chiral fermions in its long-distance spectrum

We applied a version of this theorem that does not assume:

- fermionic fields as the fields of the path integral
- the number of DOF per site
- the interactions of the (would-be) fermions

and showed the modified Villain model produces chiral fermions with no doublers, at the expense of losing locality

Future directions? 3d?

Thank you!