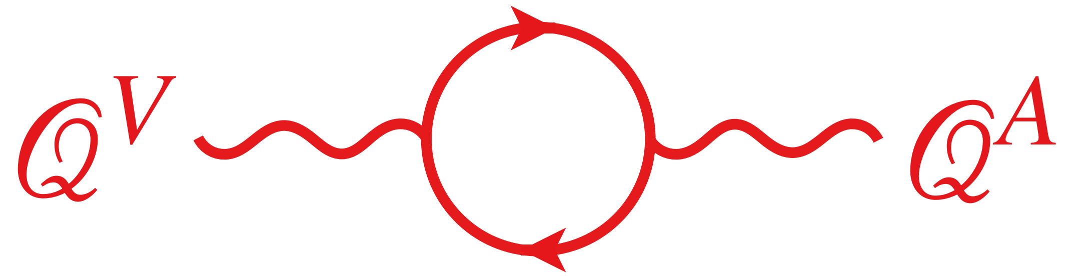
Chiral symmetries and Anomalies from Lattice Fermions

$[Q^V, Q^A] \neq 0$



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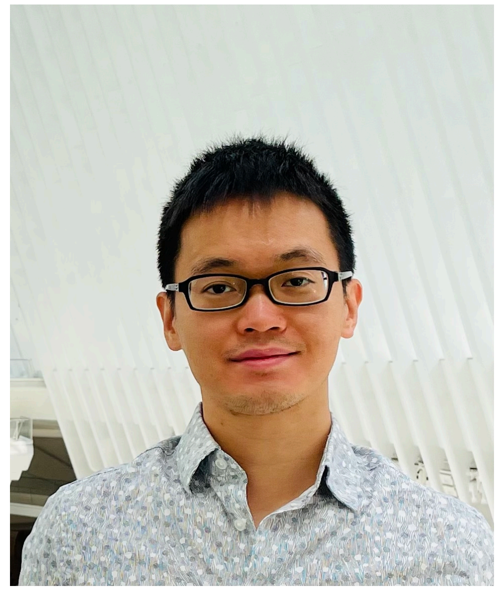
New Developments in Lattice Fermions

Syracuse University

Aug 12, 2026

Sal Pace (IAS)



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AC—Pace—Shao 2409.12220
[PRL Jan 2025]

Motivation

Fermions on the Lattice

- Long-standing challenge: how to realize chiral global symmetries of fermions on the lattice? Nielsen–Ninomiya (1981)
- Various approaches have been studied over the years:
    - Staggered fermions Kogut-Susskind (1975) Banks et al. (1976)
    - Non-local models with modified lattice derivatives Drell et al. (1976)
    - Domain wall fermions, coupled to SPT bulk Kaplan hep-lat/9206013
    - Symmetric mass generation Wang–Wen 1807.05998
    - (Modified) Villain models Seifnashri 2601.14359 Thorngren et al.2601.04304

    … Baig et al. 2607.09935 Lu–Shao 2608.02728 Dharanikota–Fidkowski 2608.02722

Anomalies

- Anomalies lead to non-perturbative constraints on infrared (IR) dynamics
  $\therefore$ extremely powerful in analyzing strongly coupled QFT/quantum matter

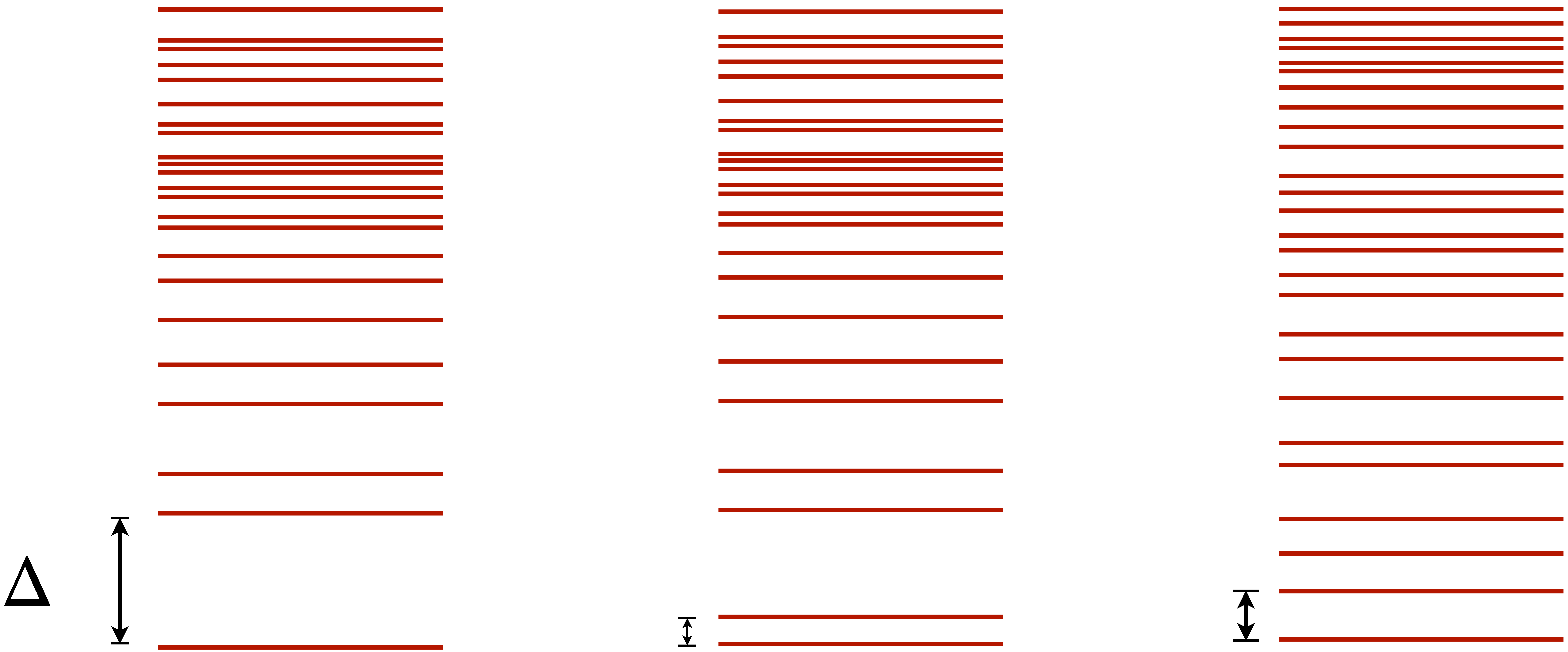
  - Obstruction to trivial gapped phases — Lieb–Schultz–Mattis (1961)

  - Sometimes, anomaly-enforced *gaplessness* — Wang–Senthil 1401.1142, Córdova–Ohmori 1910.04962

- A symmetry of a quantum theory is anomalous if it forbids a symmetric gapped phase with unique ground state

- **Anomaly matching**: anomalies of UV (ultraviolet) symmetries must be matched by anomalous IR symmetries — 't Hooft (1980)

Anomaly = guarantee of interesting IR dynamics



$\Delta = O(1)$

Gapped with unique ground state

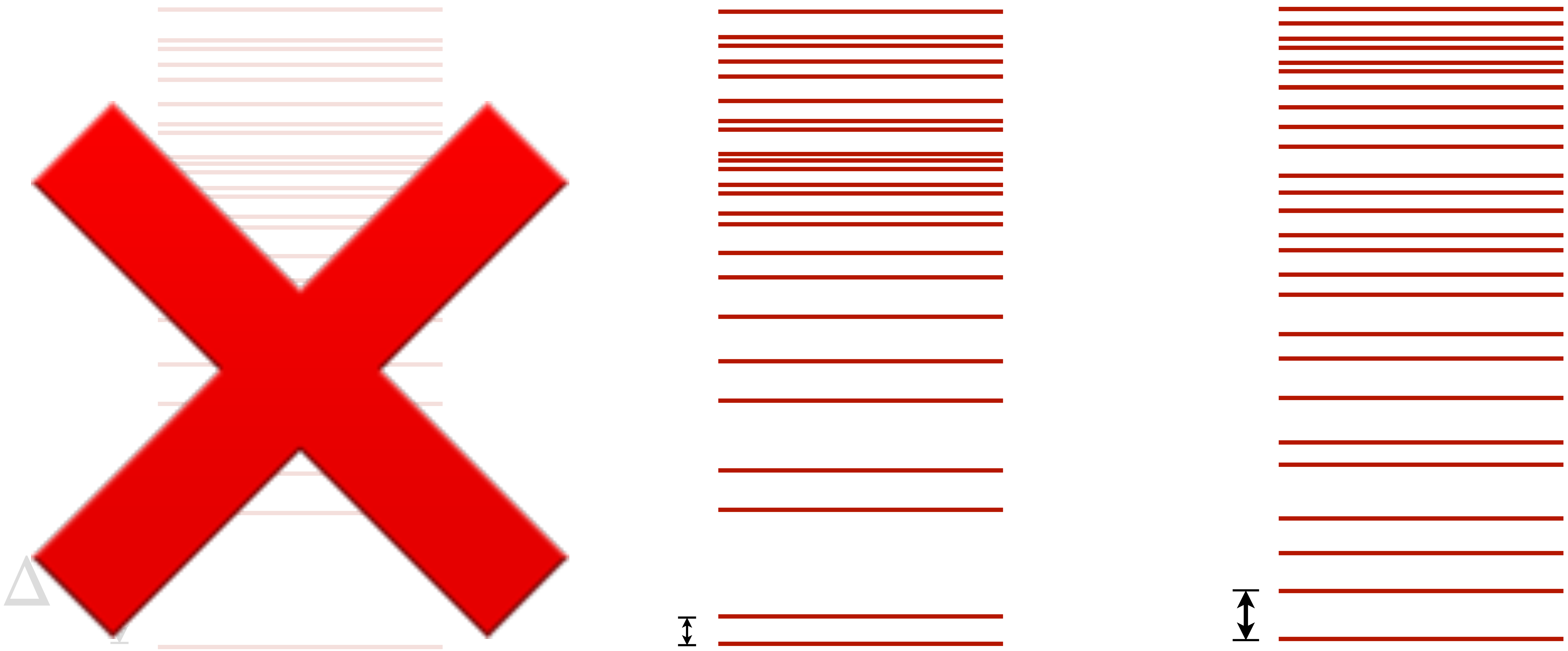
$\Delta = O(e^{-L})$

Gapped with ground state degeneracy

$\Delta = O(L^{-\alpha})$

Gapless

Anomaly = guarantee of interesting IR dynamics

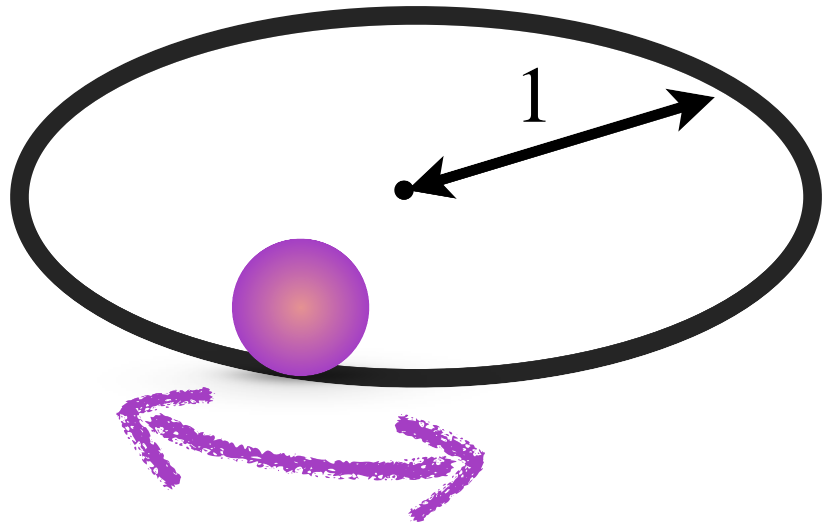


$\Delta = O(1)$	$\Delta = O(e^{-L})$	$\Delta = O(L^{-\alpha})$
Gapped with unique ground state	**Gapped with ground state degeneracy**	**Gapless**

Warm-up: anomalies in 0+1d

(a.k.a. projective reps in QM)

Particle on a ring



$$L = \frac{1}{2}\dot{x}^2 + \frac{\theta}{2\pi}\dot{x}\,,\ \ x \sim x + 2\pi$$

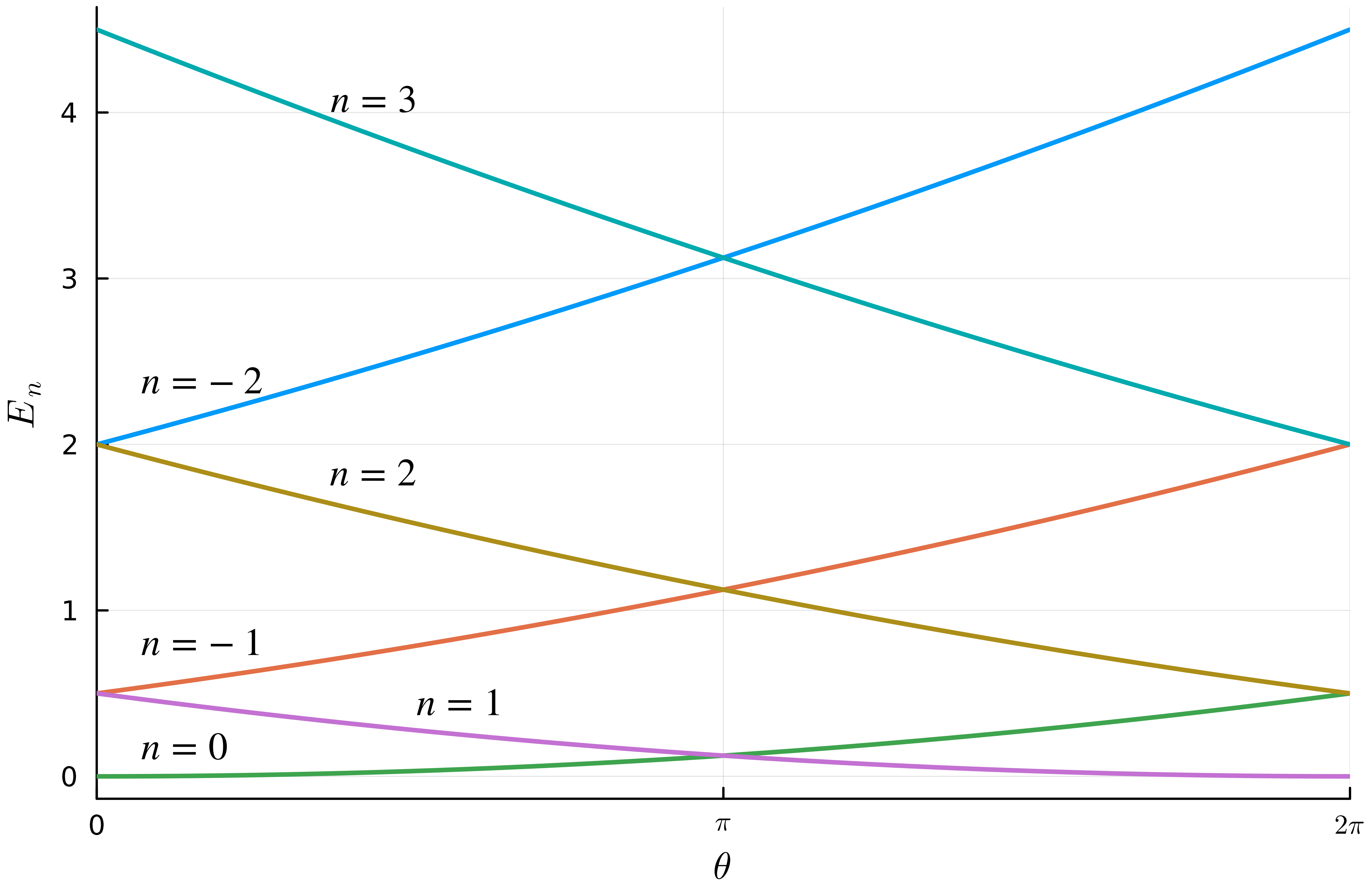
The corresponding Hamiltonian is $H = \frac{1}{2}\left(-i\frac{\partial}{\partial x} - \frac{\theta}{2\pi}\right)^2$

Its eigenfunctions are $\psi_n(x) \propto e^{inx}$, $n \in \mathbb{Z}$, with eigenvalues $E_n = \frac{1}{2}\left(n - \frac{\theta}{2\pi}\right)^2$

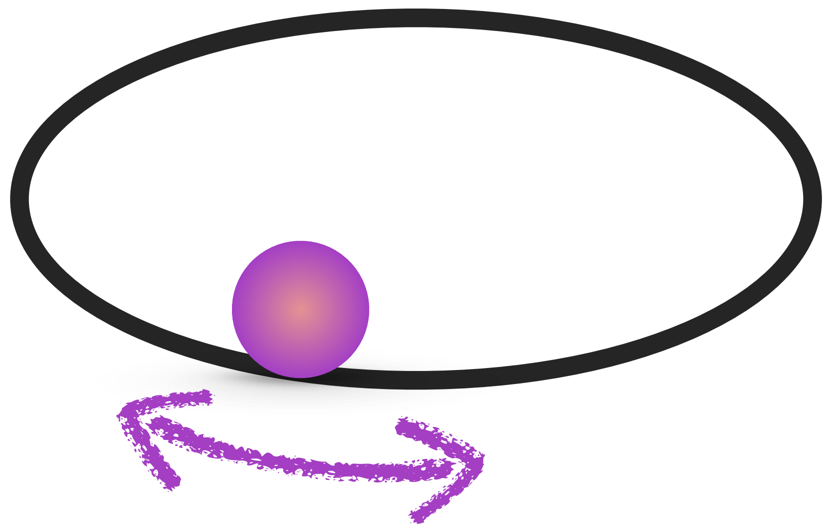
<u>Note:</u> only the value of $\theta \bmod 2\pi$ is physical, since $H_\theta = e^{-ix}H_{\theta+2\pi}e^{ix}$

e.g. App D.1 of Gaiotto et al. 1703.00501

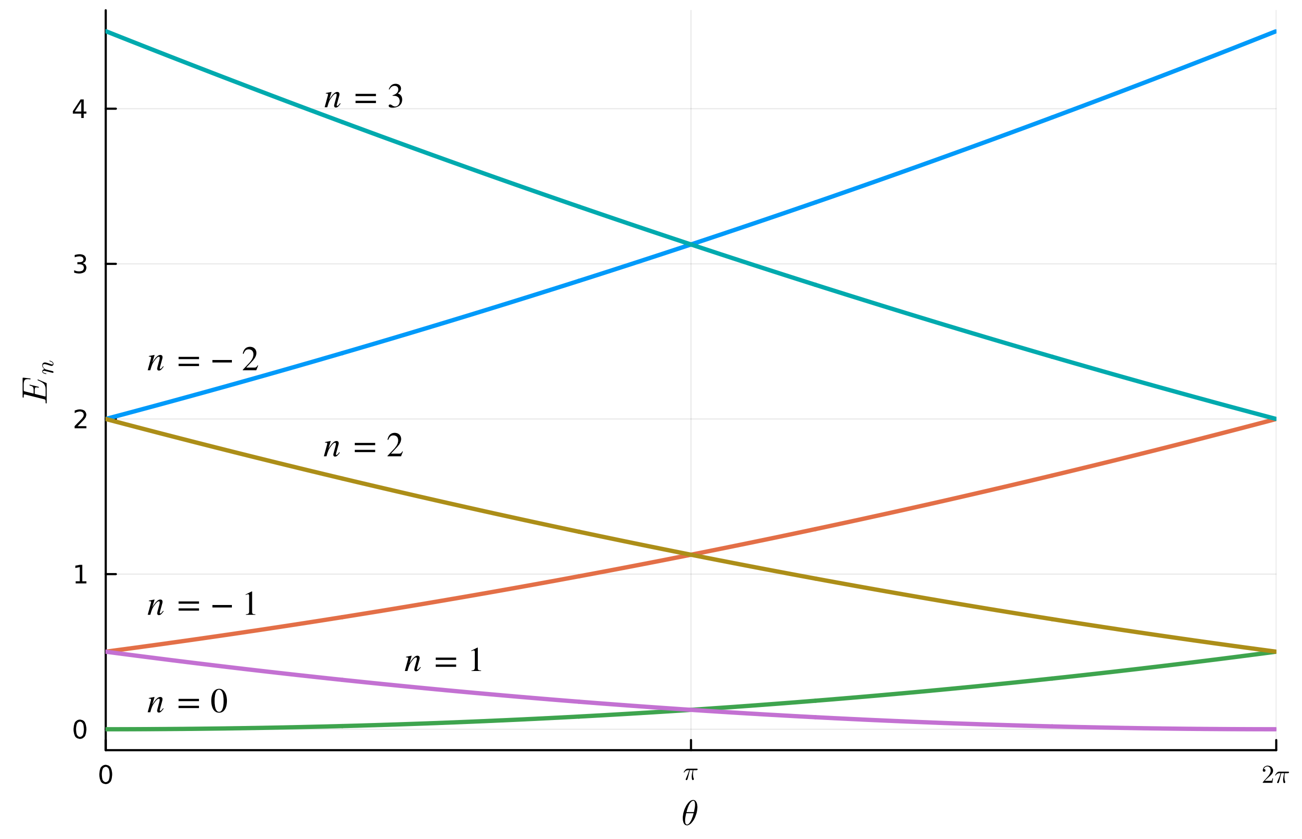
Particle on a ring



Particle on a ring



$$H = \left(-i\frac{\partial}{\partial x} - \frac{\theta}{2\pi}\right)^2$$



Degeneracies at $\theta = \pi$ are the consequence of an anomaly!

- $\mathrm{U}(1)$ shift symmetry $x \mapsto x + a$, $a \sim a + 2\pi$ for all θ $\qquad \psi_n(x) \mapsto e^{ina}\psi_n(x)$
- $\mathbb{Z}_2$ reflection symmetry $x \mapsto -x$ for $\theta = 0, \pi$ $\qquad \psi_n(x) \mapsto \begin{cases} \psi_{-n}(x) & \theta = 0 \\ \psi_{1-n}(x) & \theta = \pi \end{cases}$

$\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup has a mixed anomaly at $\theta = \pi$

- each eigenspace carries projective rep $\Rightarrow$ at least 2-fold degenerate
- symmetric potentials e.g. $V(x) = \lambda \cos(2x)$ cannot open a gap

Chiral anomaly in 1+1d

Massless free Dirac fermions

- A Dirac field forms a reducible representation of the 1+1d Clifford algebra, $\Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}$ where Ψ_L and Ψ_R are 1-component (*complex*) Weyl fermions

- Massless free Dirac field theory is described by the action

$$S = i \int dtdx \left(\Psi_L^\dagger(\partial_t - \partial_x)\Psi_L + \Psi_R^\dagger(\partial_t + \partial_x)\Psi_R \right)$$

- Ψ_L, Ψ_R : left- and right-moving Weyl fermions, respectively

Symmetries of the field theory

$$S = i \int dtdx \left(\Psi_L^\dagger (\partial_t - \partial_x) \Psi_L + \Psi_R^\dagger (\partial_t + \partial_x) \Psi_R \right)$$

- Number conservation and charge conjugation for Ψ_L and Ψ_R:
$$\left(\mathrm{U}(1)^L \rtimes \mathbb{Z}_2^{\mathscr{C}^L} \right) \times \left(\mathrm{U}(1)^R \rtimes \mathbb{Z}_2^{\mathscr{C}^R} \right) \cong \mathrm{O}(2)^L \times \mathrm{O}(2)^R$$

- $\mathrm{U}(1)^L : \Psi_L \to e^{-i\theta} \Psi_L\,, \; \Psi_R \to \Psi_R\,, \quad \mathbb{Z}_2^{\mathscr{C}^L} : \Psi_L \leftrightarrow \Psi_L^\dagger\,, \; \Psi_R \to \Psi_R$
$\mathrm{U}(1)^R : \Psi_L \to \Psi_L\,, \; \Psi_R \to e^{-i\theta} \Psi_R\,, \quad \mathbb{Z}_2^{\mathscr{C}^R} : \Psi_L \to \Psi_L\,, \; \Psi_R \leftrightarrow \Psi_R^\dagger$

- Conserved charges: $\mathscr{Q}^L = \int dx \Psi_L^\dagger \Psi_L, \quad \mathscr{Q}^R = \int dx \Psi_R^\dagger \Psi_R$

Vector and Axial $\mathrm{U}(1)$

$$S = i\int dtdx\, \left(\Psi_L^\dagger(\partial_t - \partial_x)\Psi_L + \Psi_R^\dagger(\partial_t + \partial_x)\Psi_R\right)$$

Define *vector* and *axial* charges: $\mathcal{Q}^V = \mathcal{Q}^L + \mathcal{Q}^R\,,\ \mathcal{Q}^A = \mathcal{Q}^L - \mathcal{Q}^R$

$$\mathrm{U}(1)^{\mathscr{V}} : \Psi_L \to e^{-i\theta}\Psi_L\,,\ \Psi_R \to e^{-i\theta}\Psi_R$$
$$\mathrm{U}(1)^{\mathscr{A}} : \Psi_L \to e^{-i\theta}\Psi_L\,,\ \Psi_R \to e^{+i\theta}\Psi_R$$

These symmetries have a mixed anomaly, called the “chiral anomaly”

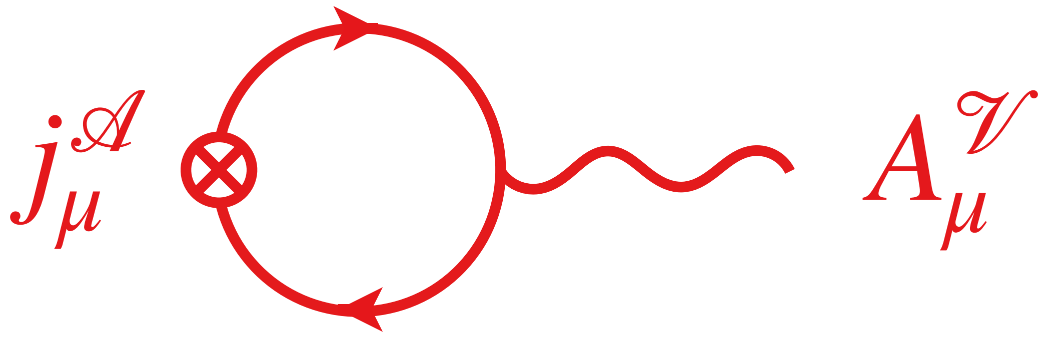
Schwinger (1959)

Chiral anomaly

- $\mathrm{U}(1)^{\mathscr{A}}$ symmetry is broken upon gauging $\mathrm{U}(1)^{\mathscr{V}}$ symmetry

Johnson (1963)

$$\langle \partial^\mu j_\mu^{\mathscr{A}} \rangle = \frac{1}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu}^{\mathscr{V}}$$



- Relatedly, the equal-time commutator of the two current densities has a contact term (*“Schwinger term”*)

$$[j_t^{\mathscr{V}}(t, x_1), j_t^{\mathscr{A}}(t, x_2)] \propto i\delta'(x_1 - x_2)$$

Schwinger (1959)

- Impossible to realize such a commutator in a finite-dimensional H.S.
(∵ *l.h.s. has vanishing trace but r.h.s. does not*)

Kapustin–Sopenko 2401.02533

- Realized recently in modified Villain models with *infinite-dimensional H.S.*

Seifnashri 2601.14359   Lu–Shao 2608.02728   Dharanikota–Fidkowski 2608.02722

See also: Berkowitz et al. 2310.17539

A lattice axial $\mathrm{U}(1)$ symmetry

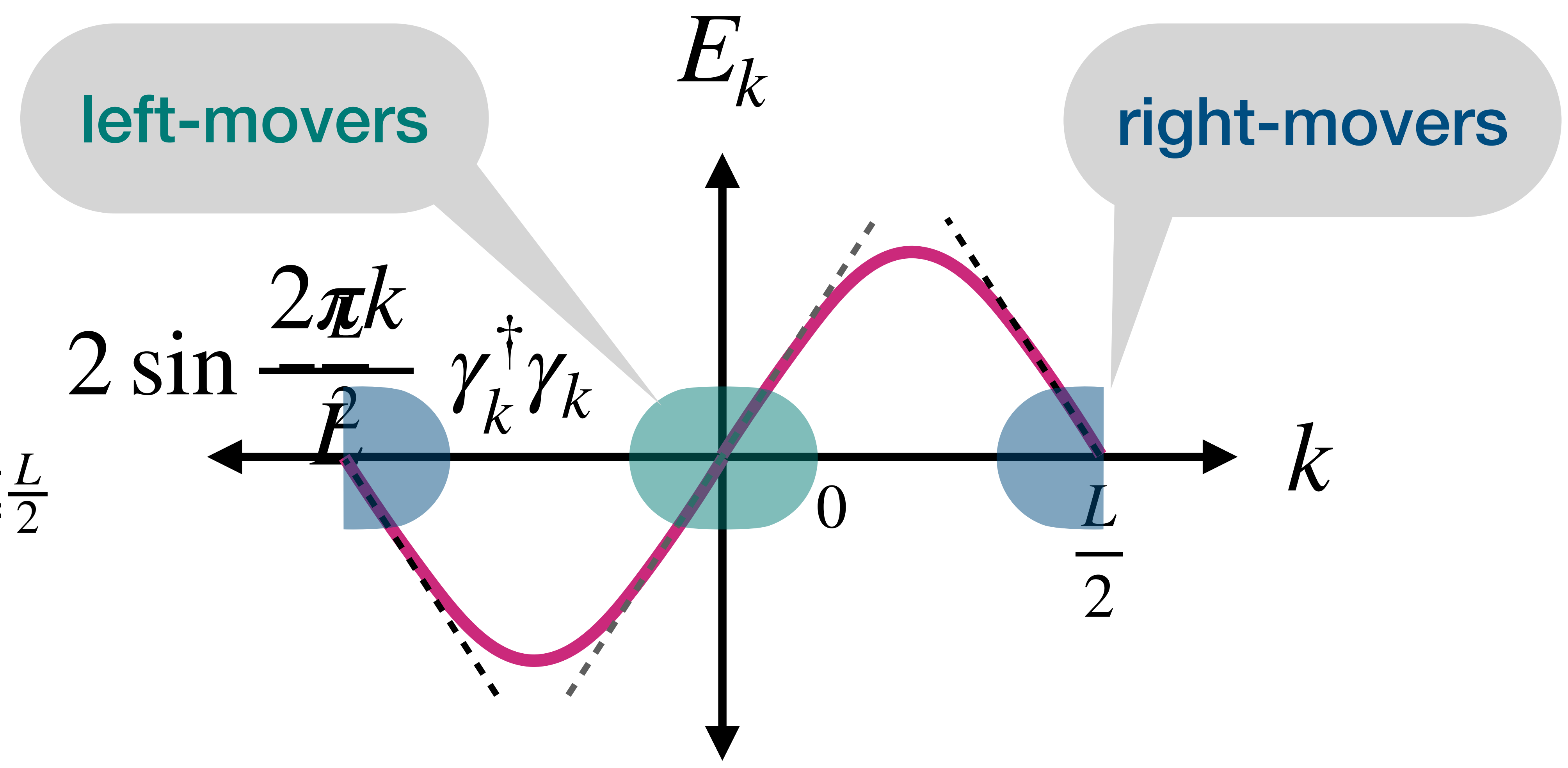
Staggered fermions in 1+1d

$$H_f = -i \sum_{j=1}^{L} \left(c_j^\dagger c_{j+1} + c_j c_{j+1}^\dagger \right)$$

Kogut-Susskind (1975)

- Single-component complex fermions on a periodic chain of (*even*) L sites
- Intuitively, two components of continuum Dirac field are distributed in a "staggered" manner between even and odd sites

$$\gamma_k = \frac{1}{\sqrt{L}} \sum_{j=1}^{L} e^{-i\frac{2\pi kj}{L}} c_j \quad \Rightarrow \quad H_f = \sum_{-\frac{L}{2}<k\leq\frac{L}{2}} 2\sin\frac{2\pi k}{L}\, \gamma_k^\dagger \gamma_k$$



Staggered fermions in 1+1d

$$H_f = -i\sum_{j=1}^{L}\left(c_j^\dagger c_{j+1} + c_j c_{j+1}^\dagger\right)$$

- Single-component complex fermions on a periodic chain of (*even*) L sites

- Intuitively, two components of continuum Dirac field are distributed in a "staggered" manner between even and odd sites

- H_f has the on-site symmetry $c_j \to e^{-i\theta}c_j$, generated by $Q^V = \sum_{j=1}^{L}(c_j^\dagger c_j - \frac{1}{2})$,
  $\Rightarrow$ lattice vector $\mathrm{U}(1)$ symmetry

- **Question:** Does H_f also have a lattice axial $\mathrm{U}(1)$ symmetry?

Staggered fermions in 1+1d

$$H_f = -i \sum_{j=1}^{L} \left(c_j^\dagger c_{j+1} + c_j c_{j+1}^\dagger \right)$$

- Single-component complex fermions on a periodic chain of (*even*) L sites
- Intuitively, two components of continuum Dirac field are distributed in a "staggered" manner between even and odd sites
- H_f has the on-site symmetry $c_j \to e^{-i\alpha} c_j$, generated by $Q^V = \sum_{j=1}^{L} (c_j^\dagger c_j - \frac{1}{2})$,
  $\Rightarrow$ lattice vector $\mathrm{U}(1)$ symmetry
- **Question:** Does H_f also have a lattice axial $\mathrm{U}(1)$ symmetry?

Answer: **Yes!**

Hint: decompose each *complex* fermion into two *real* fermions

Be Real

- Recall, $\mathscr{Q}^A = \mathscr{C}^R \mathscr{Q}^V (\mathscr{C}^R)^{-1}$, where $\mathscr{C}^R \, \Psi_R (\mathscr{C}^R)^{-1} = \Psi_R^\dagger$

$$\boxed{\begin{array}{l} \mathscr{Q}^V = \mathscr{Q}^L + \mathscr{Q}^R \\ \mathscr{Q}^A = \mathscr{Q}^L - \mathscr{Q}^R \end{array}}$$

- To find UV axial charge, look for a UV symmetry that flows to $\mathscr{C}^R$ in the IR

- Useful to decompose Weyl fermion into Majorana-Weyl (*real*) fermions as $\Psi_R = A_R + iB_R$, so that $\mathscr{C}^R : A_R \to A_R \, , \, B_R \to \; - B_R$

- Decompose the lattice fermions similarly into real fermions, i.e. $c_j = \dfrac{a_j + ib_j}{2}$ with $\{a_j, a_{j'}\} = \{b_j, b_{j'}\} = 2\delta_{j,j'} \, , \{a_j, b_{j'}\} = 0$

Majorana translation T_b

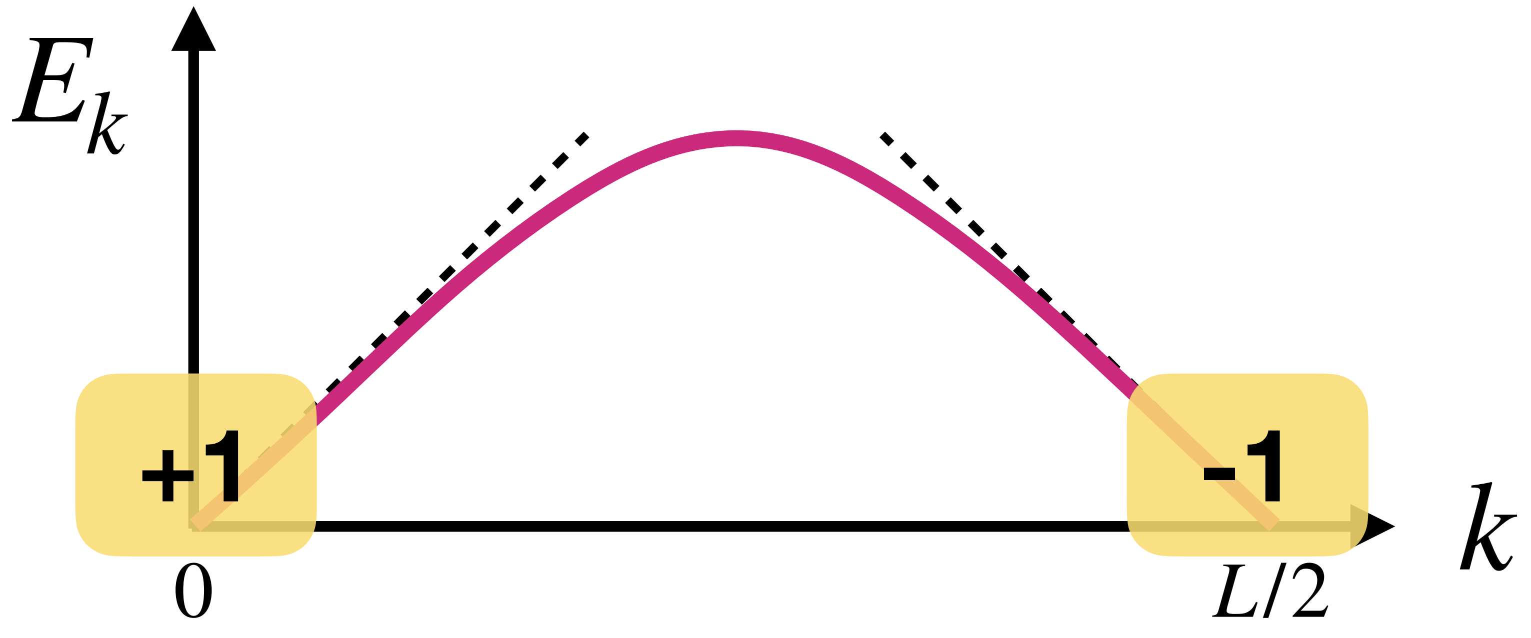
- In real fermion representation, $H_f = -\frac{i}{2}\sum_j (a_j a_{j+1} + b_j b_{j+1})$
- Using momentum modes, $H_f = \sum_{0<k<L/2} \sin\frac{2\pi k}{L}(\alpha_{-k}\alpha_k + \beta_{-k}\beta_k)$

$$\boxed{\alpha_k = \frac{1}{\sqrt{L}}\sum_{j=1}^{L} e^{-i\frac{2\pi kj}{L}} a_j}$$

- α_k, β_k with $k \approx 0$ and $L/2$ identified with continuum fields A_L, B_L and A_R, B_R in the IR

- Majorana translation, $T_b : b_j \to b_{j+1}$ acts on these momentum modes as

$$T_b\,\beta_{-k}|\Omega\rangle = e^{-i\frac{2\pi k}{L}}\beta_{-k}|\Omega\rangle$$



Axial $\mathrm{U}(1)$ on the lattice

- Conclusion: T_b flows to $\mathscr{C}^R$ in the IR limit

- Define $Q^A = T_b Q^V T_b^{-1}$ in analogy with the continuum $\mathcal{Q}^A = \mathscr{C}^R \mathcal{Q}^V (\mathscr{C}^R)^{-1}$

$$\boxed{Q^V = \sum_j \frac{i}{2} a_j b_j \qquad Q^A = \sum_j \frac{i}{2} a_j b_{j+1}}$$

cf. "arrow charge" of Thacker (1994), Horvath-Thacker (1998)

- Q^A has several nice properties:
  - commutes with H_f,
  - has quantized eigenvalues,
  - action on fermions is locality-preserving,
  - is a sum of mutually-commuting local fermion bilinears $\Rightarrow$ $e^{i\theta Q^A} = \prod_j e^{i\theta q^A_{j,j+1}}$

$$\boxed{\begin{aligned} e^{i\theta Q^A} a_j e^{-i\theta Q^A} &= \cos\theta\, a_j + \sin\theta\, b_{j+1} \\ e^{i\theta Q^A} b_j e^{-i\theta Q^A} &= \cos\theta\, b_j - \sin\theta\, a_{j-1} \end{aligned}}$$

A comment on Nielsen-Ninomiya

It follows from the Nielsen-Ninomiya theorem, that a lattice axial charge cannot satisfy both of the following properties:

(1) its eigenvalues are integer-quantized, and

(2) it commutes with the vector charge

Nielsen-Ninomiya (1981)

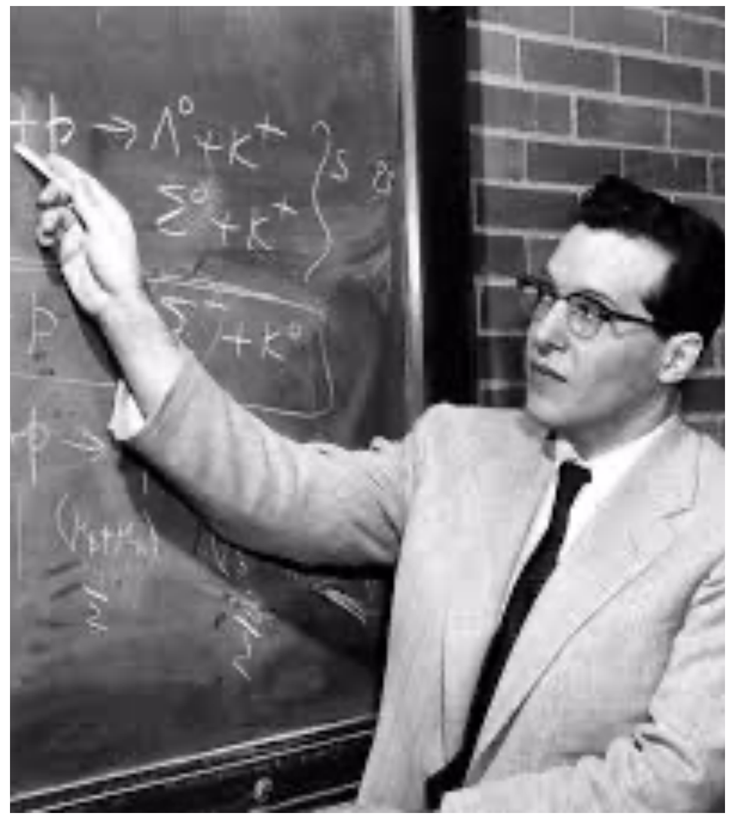
Friedan (1982)

- Q^A satisfies (1) but not (2), but one can construct another charge $\tilde{Q}^A$ that satisfies (2) but not (1):

$$Q^A = \sum_j \frac{i}{2} a_j b_{j+1} \qquad \tilde{Q}^A = \sum_j \frac{i}{4}(a_j b_{j+1} - b_j a_{j+1})$$

Banks—Susskind—Kogut (1976)

- In the IR, both Q^A and $\tilde{Q}^A$ flow to $\mathcal{Q}^A$



Schwinger’s anomaly from Onsager’s algebra

$Q^V, Q^A \implies$ Onsager algebra

- Unlike their continuum counterparts, the lattice axial and vector charges **do not** commute with each other, $[Q^V, Q^A] \neq 0$
- In fact, they generate the Onsager algebra! Onsager (1944)
- Define $Q_n = \frac{i}{2}\sum_j a_j b_{j+n}$ and $G_n = \frac{i}{2}\sum_j (a_j a_{j+n} - b_j b_{j+n})$

$$\boxed{\begin{aligned} Q^V &= Q_0 \\ Q^A &= Q_1 \end{aligned}}$$

- They satisfy the commutation relations,

$$[Q_n, Q_m] = iG_{m-n}\,, \qquad [Q_n, G_m] = 2i(Q_{n-m} - Q_{n+m})\,, \qquad [G_n, G_m] = 0$$

Q_0 **and** Q_1 **satisfy the Dolan-Grady relations** $[Q_0, [Q_0, [Q_0, Q_1]]] = 4[Q_0, Q_1]$

Emanant axial and vector $\mathrm{U}(1)$ symmetries

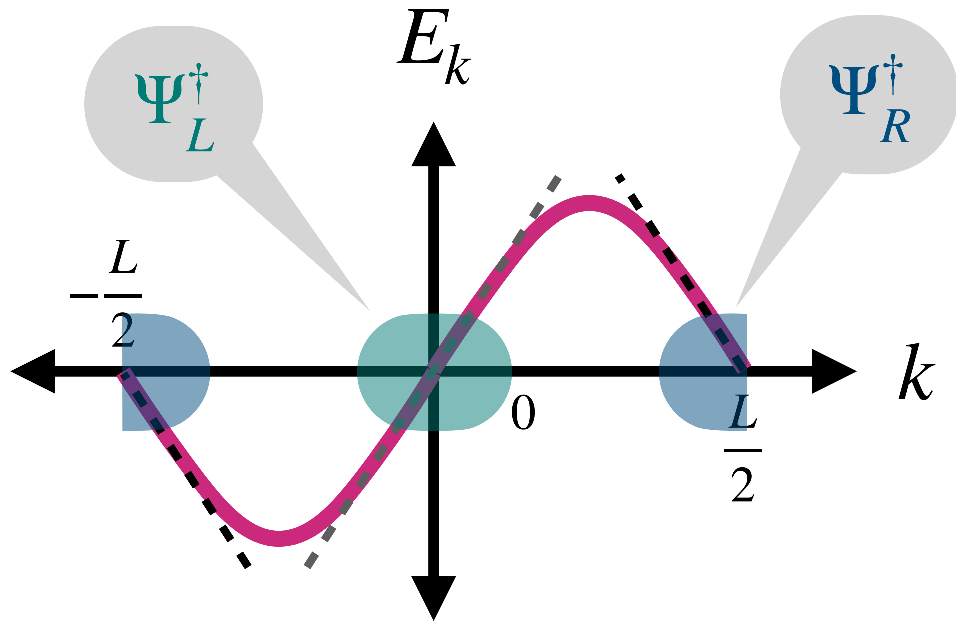
Cheng-Seiberg 2211.12543

How to see that the IR symmetries emanate from the UV symmetries?

- Recall the momentum modes, $\gamma_k^\dagger = \frac{1}{\sqrt{L}} \sum_j e^{i\frac{2\pi kj}{L}} c_j^\dagger$
- The low-energy modes near $k \approx 0$ and $L/2$ transform under the symmetry charges in a way that mirrors the IR field theory

$$[Q^V, \gamma^\dagger_{k\approx 0,\frac{L}{2}}] = \gamma^\dagger_{k\approx 0,\frac{L}{2}} \qquad [\mathcal{Q}^V, \Psi^\dagger_{L,R}] = \Psi^\dagger_{L,R}$$

$$[Q^A, \gamma^\dagger_{k\approx 0,\frac{L}{2}}] \approx \pm\, \gamma^\dagger_{k\approx 0,\frac{L}{2}} \qquad [\mathcal{Q}^A, \Psi^\dagger_{L,R}] = \pm\, \Psi^\dagger_{L,R}$$



For fixed n as $L \to \infty$, we have $Q_n \to \begin{cases} \mathcal{Q}^A & n \text{ odd}, \\ \mathcal{Q}^V & n \text{ even}, \end{cases}$ and $G_n \to 0$.

Emanant axial and vector $\mathrm{U}(1)$ symmetries

The lattice Q^V and Q^A flow to the continuum $\mathcal{Q}^V$ and $\mathcal{Q}^A$, i.e. the chiral anomaly is *emanant*, rather than *emergent*!

Anomaly checklist 📋

1. Anomaly matching?

2. Obstruction to gauging the symmetry?

3. Anomaly inflow from 2+1d SPT (*symmetry protected topological*) phase?

4. Anomaly cancellation by stacking?

$Q^V, Q^A \implies$ no symmetric gapped phase

- Can show that any Q^V and Q^A-symmetric *local* Hamiltonian is **gapless**

- Family of symmetric Hamiltonians is parameterized as

$$H_{\text{symm}}(\{g_n\}) = \sum_{j=1}^{L} \sum_{n} g_n(ic_j^\dagger c_{j+n} + \mathrm{h.c.})$$

- It is possible to open an energy gap with deformations that preserve Q^V while breaking Q^A (*and vice-versa*)

- Parallels the consequences of the chiral anomaly in the IR

Cf. Frishman et al. (1981)

Anomaly checklist 📋

1. Anomaly matching? ✅

2. Obstruction to gauging the symmetry?

3. Anomaly inflow from 2+1d SPT (*symmetry protected topological*) phase?

4. Anomaly cancellation by stacking?

Coupling to gauge fields

Upon gauging $\mathrm{U}(1)^V$: Q^A is not gauge-invariant, $\tilde{Q}^A(U)$ is not conserved

- The gauged Hamiltonian is $H_V = -\sum_j \left(i\, c_j^\dagger U_{j+\frac{1}{2}} c_{j+1} + \mathrm{h.c.} \right) + K \sum_j L^2_{j+\frac{1}{2}}$

  where $[L_{j+\frac{1}{2}}, U_{j'+\frac{1}{2}}] = \delta_{j,j'} U_{j'+\frac{1}{2}}$, with Gauss constraints, $L_{j+\frac{1}{2}} - L_{j-\frac{1}{2}} = \frac{i}{2} a_j b_j + \frac{(-1)^j}{2}$

- The gauge-invariant version of $\tilde{Q}^A$ is $\tilde{Q}^A(U) = \frac{1}{2} \sum_j \left(c_j^\dagger U_{j+\frac{1}{2}} c_{j+1} - c_j U^\dagger_{j+\frac{1}{2}} c^\dagger_{j+1} \right)$

$$-i\,[\tilde{Q}^A(U), H_V] = -\frac{K}{2} \sum_j \{L_{j+\frac{1}{2}}, h_{j+\frac{1}{2}}\} \neq 0$$

Gauging $\mathrm{U}(1)^A$

- To further illustrate the fact that U(1)^A is anomaly-free, note that it can be gauged on the lattice:

$$H_A = \sum_j \left[-\frac{i}{4}(a_j - ib_{j+1})U_{j+\frac{1}{2}}(a_{j+1} + ib_{j+2}) + \mathrm{h.c.} \right] + K \sum_j L^2_{j+\frac{1}{2}}$$

$[L_{j+\frac{1}{2}}, U_{j'+\frac{1}{2}}] = \delta_{j,j'} U_{j'+\frac{1}{2}}$, and the Gauss law is $L_{j+\frac{1}{2}} - L_{j-\frac{1}{2}} = \frac{i}{2} a_j b_{j+1} + \frac{(-1)^j}{2}$

- Unitarily equivalent to the conventional lattice Schwinger model obtained by gauging $\mathrm{U}(1)^V$:

$$H_V = -\sum_j \left[i\, c_j^\dagger U_{j+\frac{1}{2}} c_{j+1} + \mathrm{h.c.} \right] + K \sum_j L^2_{j+\frac{1}{2}}$$

Anomaly checklist 📋

1. Anomaly matching? ✅ 🤔
2. Obstruction to gauging the symmetry? ✅
3. Anomaly inflow from 2+1d SPT (*symmetry protected topological*) phase?
4. Anomaly cancellation by stacking?

Some remarks

- A lattice avatar of the Schwinger term:

$$[q_j^V, \tilde{q}^A_{j'+1/2}] = \frac{i}{2}\sum_j (\delta_{j,j'} - \delta_{j-1,j'}) h_{j'+1/2}$$

- Interactions? Lattice version of the 4-fermion Thirring coupling $(\overline{\Psi}\gamma^\mu\Psi)^2$ preserves $\mathrm{U}(1)^V$ but breaks $\mathrm{U}(1)^A$

- $\mathrm{U}(1)^L$ and $\mathrm{U}(1)^R$ on the lattice? $Q^V \pm Q^A$ have unquantized eigenvalues; do not generate locality-preserving symmetry actions

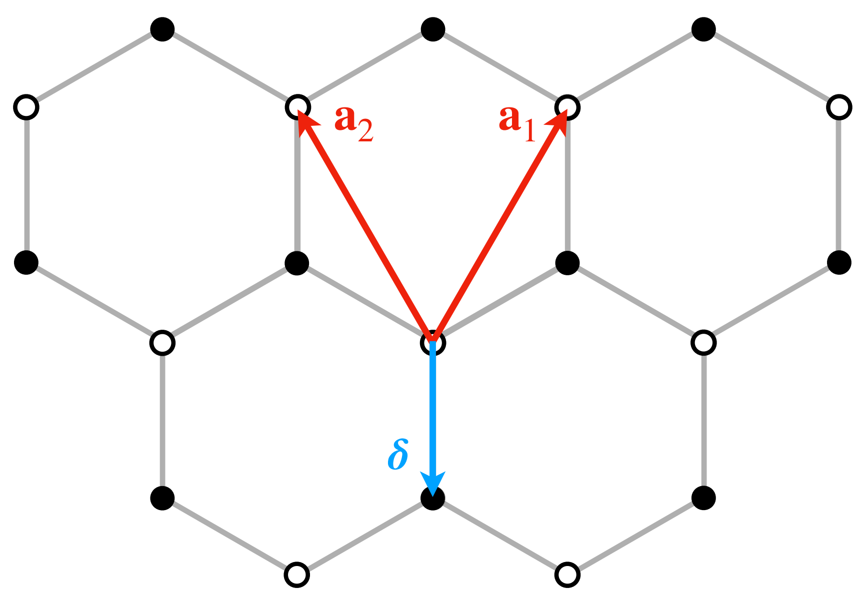
cf. Lu–Shao 2608.02728

More examples

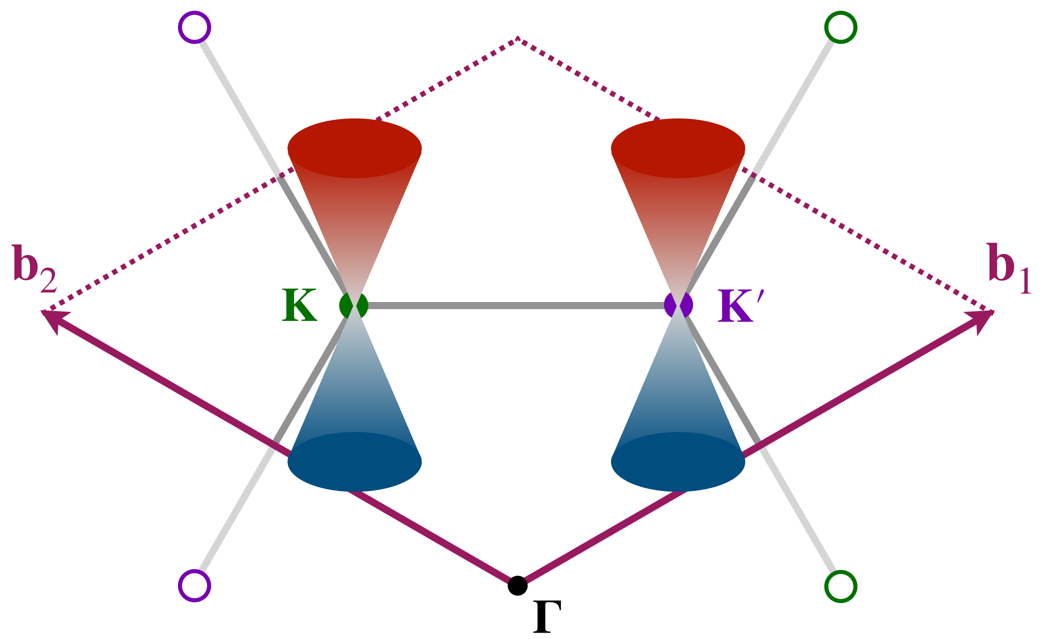
Other anomalies arising from Onsager-like symmetries:

- $\mathrm{SU}(2)$ "parity" anomaly in 2+1d   Pace–Kim–**AC**–Shao 2505.04684

$\mathrm{Ons}_2 \rtimes \{\mathsf{R}_i, \mathsf{T}\}$



$\longrightarrow$



$Z_X[A,g] \xrightarrow{\mathcal{T}} e^{-i\pi\eta[A,g]} Z_X[A,g]$

- Witten's $\mathrm{SU}(2)$ anomaly in 3+1d   Gioia–Thorngren 2503.07708

Also: Catterall et al. 2501.10862, Kim et al. 2512.04150

Summary and Outlook

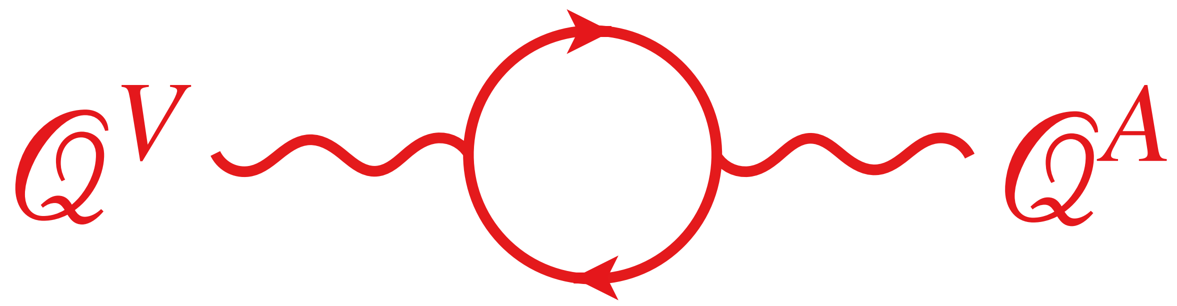
- We discussed how symmetries of 1+1d massless Dirac fermions are realized exactly in a lattice Hamiltonian

- The chiral anomaly emanates from a lattice Onsager symmetry: first in a family of such examples

- Classification of anomalies of Onsager symmetries?

- Non-Abelian perturbative anomalies on the lattice?

$$[Q^V, Q^A] \neq 0 \longrightarrow$$



Thank You

Backup Slides

Sketch of gaplessness proof

(1) Requiring invariance under both $\mathrm{U}(1)^V$ and $\mathrm{U}(1)^A$ implies invariance under $e^{-i\frac{\pi}{2}Q^A}e^{i\frac{\pi}{2}Q^V}$, which acts identically to $T_b T_a^{-1}$

(2) Imposing $T_b T_a^{-1}$ rules out any coupling between a_j and b_j Majoranas, if the Hamiltonian must be a sum of local terms

(3) Any quartic or higher order coupling in either a_j or b_j breaks $\mathrm{U}(1)^V$

(4) The only allowed local Hamiltonian terms are of the form $\sum_j (ia_j a_{j+n} + ib_j b_{j+n})$

(5) The entire family of such Hamiltonians is gapless $\sum_n g_n \sum_j (ic_j^\dagger c_{j+n} + h.c.)$

Lattice T-duality

- Bosonizing the staggered fermion Hamiltonian gives the spin-1/2 XX model

$$H_{\mathrm{XX}} = \sum_{j=1}^{L} \left(X_j X_{j+1} + Y_j Y_{j+1} \right)$$

- The IR limit of this Hamiltonian is the $c = 1$ *compact boson CFT* at radius $R = \sqrt{2}$ (*in our conventions, $R = 1$ is the self-dual radius*)

$$S = \frac{R^2}{4\pi} \int d\Phi \wedge \star\, (d\Phi) = \frac{1}{4\pi R^2} \int d\Theta \wedge \star\, (d\Theta)$$

- We find two U(1) symmetries of the XX model that flow to the “momentum” and “winding” U(1) symmetries of the CFT

$$Q^M = \frac{1}{2} \sum_{j=1}^{L} Z_j \qquad Q^W = \frac{1}{4} \sum_{n=1}^{L/2} (X_{2n-1} Y_{2n} - Y_{2n} X_{2n+1})$$

- The T-duality $R \to 1/R$ of the CFT is realized as a unitary on the lattice

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