Kähler Fermions and Symmetric Mass Generation

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Takeaways

1. Nielsen Ninimoiya theorem - lattice theory cannot be anomalous, gravitational anomaly in Euclidean Kähler-Dirac fermion (free-massless) is an exception.
2. Kähler-Dirac fermions satisfy Dirac equation describing $2^{D/2}$ degenerate flavors of Dirac fermions in flat spacetime.
3. Discrete Kähler-Dirac fermions are equivalent to staggered fermions in flat space describing $2^{D/2}$ degenerate flavors of Dirac fermions.
4. Mod 2 anomaly.

Anomalies in the 80s

1. ABJ
   - a global symmetry of the classical theory breaks when quantized.
2. Gauge Invariance
   - redundancy in quantized theory due to local (gauge) symmetry.
   - must be canceled to have a consistent theory (Standard Model).
3. 't Hooft anomalies
   - coupling a global symmetry of the quantized theory to a **background** gauge field yield.
   - provides information on non-perturbative structure of theory from its global symmetries.
   - This is RG invariant. If non-zero in U.V. $\rightarrow$ I.R. cannot be trivial (eg. massless states, SSB, TQFT)

Anomalies in the modern era, post 2015

1. Discrete symmetries can be anomalous - Time Reversal.
2. Anomalies arising from higher form symmetries which act on extended objects - Center Symmetry
3. Anomalies from symmetries without an inverse operation - Non-invertible symmetry
4. Can build consistent theories (with anomalies) in D dimensions by adding a $D+1$ dimensional bulk theory - Anomaly Inflow – SPT states and topological superconductors or DWF.

> Understand in terms of new math (bordism groups etc).

In this workshop:

> Anomalies can be found in lattice systems with finite dof !

Anomaly inflow

Partition function of massive fermions in $D+1$ dimensions is TQFT that encodes the 't Hooft anomaly for D massless fermions on boundary

- Familiar in 3D. Integrate out massive fermions $\rightarrow$ Chern-Simons action i.e Z_{3D} picks up a phase.
- Add boundary $\rightarrow$ 2D massless edge states with anomaly.
- Boundary anomaly cancels phase from 3D bulk - anomaly inflow.

Non-anomalous lattice theories

Dirac action on a Euclidean lattice:

$$S = \sum_x \bar{\psi}(x)\gamma_\mu \Delta_\mu \psi(x) = \sum_k \bar{\psi}(k) i\gamma_\mu \sin(k_\mu a)\psi(k)$$

Chiral symmetry: $\psi \to e^{i\alpha\gamma_5}\psi$, $\bar{\psi} \to e^{i\alpha\gamma_5}\bar{\psi}$

BUT zeroes of S if $k_\mu = n_\mu \left(\frac{\pi}{a}\right)$, $n_\mu = 0, 1$. fermion doubling

n_μ zero has eigenvalue $\gamma_5 = (-1)^{n_\mu}$ and $\sum_{n_\mu} \gamma_5 = 0$
Nielsen-Ninomiya thm. No lattice chiral fermions

- **Interpretation:** lattice theory (finite dof) cannot be anomalous due to fermion doubling.
- **Modern viewpoint:** onsite lattice symmetries cannot be anomalous unless they are a boundary of the bulk theory (eg. Domain Wall Fermions).

Counter example : Kähler-Dirac fermions on curved space

Onsite, exact twisted $U(1)$ lattice chiral symmetry giving rise to gravitational anomaly dependent only on the euler characteristic χ, of the manifold.

Kähler-Dirac fermions

Kähler-Dirac equation

$$(K + m)\Phi = 0 \quad \text{where} \quad K = d - d^\dagger$$

- Kähler-Dirac field $\Phi = (\phi, \phi_\mu, \phi_{\mu\nu}, \dots)$.
- Grassmann p-forms.
- Same equation for any manifold.
- No need for spin structure, spin connection, vielbein formalism

Kähler-Dirac equation

$$\int d^D x \, \sqrt{g} \, \bar{\Phi}(K + m)\Phi = 0 \quad \text{where} \quad K = d - d^\dagger$$
$$K^2 = dd^\dagger + d^\dagger d = \Box_{\text{Hodge}}$$

Connection between Landau-Kähler and Dirac

From Kähler-Dirac field $\Phi = (\phi, \phi_\mu, \dots)$ build matrix Ψ

$$\Psi = \phi I + \phi_\mu \gamma_\mu + \phi_{\mu\nu}\gamma_\mu\gamma_\nu + \dots$$

Satisfies flat space Dirac equation

$$(\partial\!\!\!/ + m)\Psi = 0$$

$2^{D/2}$ columns of Ψ are (degenerate) Dirac spinors

On curved surface, the story is different.

Triangulation

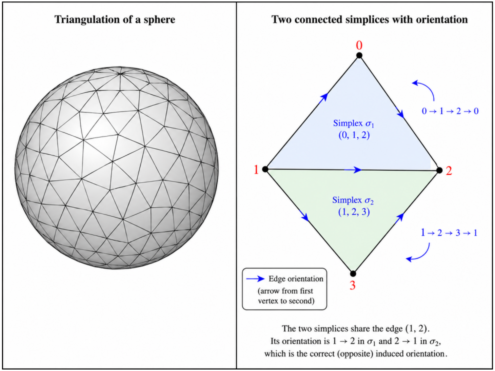


Figure: Triangulated curved surface - curved lattice

P.C. - ChatGPT

Counter example - Kähler-Dirac fermions on curved space

Oriented triangulation composed of p-simplices $C_p = [a_0 \ldots a_p]$
Boundary operator $\delta C_p = \sum_{i=0}^{p} (-1)^i [a_0 \ldots \hat{a}_i \ldots a_p] \leftarrow C_{p-1}$
fermion fields live on p-simplices eg sites, links, triangles,...
Extend δ to fields:

$$\delta\Phi(C_p) = \sum_{p-1} L(p, p-1)\Phi(C_{p-1}) \quad \text{with} \quad L(p, p-1) \in \pm 1$$

$$\boxed{(\delta - \delta^\dagger)\Phi = 0 \text{ (discrete) Kähler-Dirac eqn.}}$$

Twisted chiral symmetry $\Phi_p \to e^{i\alpha\Gamma}\Phi_p$, $\bar{\Phi}_p \to e^{i\alpha\Gamma}\bar{\Phi}_p$

where $\Gamma\Phi_p = (-1)^p \Phi_p$ and $\{\Gamma, (\delta - \delta^\dagger)\}_+ = 0$

Onsite, exact $U(1)$ lattice symmetry

A gravitational anomaly

Take triangulation of sphere.

$$Z = \int \prod_p d\bar{\Phi}_p \, d\Phi_p e^{-S_{\mathrm{KD}}(\Phi,\bar{\Phi})}$$

Under $U(1)_\Gamma$

$$Z \to \int \prod_p d\bar{\Phi}_p \, d\Phi_p \, e^{2i\alpha N_p \Gamma_p} e^{-S_{\mathrm{KD}}(\Phi,\bar{\Phi})}$$

$$= e^{2i\alpha(N_0 - N_1 + N_2 - \ldots + N_d)} Z = e^{2\chi\alpha i} Z \leftarrow \textit{anomaly}!$$

$$\boxed{\text{On sphere } \chi = 2 \; U(1) \to Z_4}$$

onsite anomalous lattice symmetry exact zero modes from spacetime topology, need 2 flavors of Kähler-Dirac fermions for anomaly cancellation when gauged.

Discrete Kähler fermions = staggered fermions

Discrete Kähler fermions are equivalent to staggered fermions in flat space.

On a square lattice,

- Matrix fermion field using Kähler-Dirac field,

$$\Psi_{\rm cont}(x) = \phi(x)I + \phi_\mu(x)\gamma_\mu + \phi_{\mu\nu}(x)\gamma_\mu\gamma_\nu + \cdots$$

- Matrix fermion field using Staggered fermion field,

$$\Psi_{stagg}(2x) = \chi(x)I + \chi(x+\mu)\gamma_\mu + \chi(x+\mu+\nu)\gamma_\mu\gamma_\nu + \cdots$$

1. χ at $x+\mu$ corresponds to $\mu^{\rm th}$ tensor component of ϕ on block lattice with twice lattice spacing.
2. staggered phases arise from antisymmetry of forms with d and $\Gamma \to \epsilon(x)$

Euclidean staggered fermion model

$$S = \sum_{x,\mu} \bar{\chi}^a(x)\eta_\mu(x)\Delta_\mu \chi^a(x) - y^2 \sum_x (\bar{\chi}^a \chi^a)^2 \quad a = 1, N$$

where $\eta_\mu(x) = (-1)^{\sum_{\nu=1}^{\mu-1} x_\nu}$.
Invariant under $SU(N)$ symmetry $\chi \to e^{i\beta}\chi \quad \bar{\chi} \to e^{-i\beta}\bar{\chi}$
Z_4 symmetry:

$$\chi(x) \to \omega^k \chi(x) \quad \text{and} \quad \bar{\chi}(x) \to \omega^k \bar{\chi}(x) \quad \omega = i\epsilon(x), k = 0 \ldots 3$$

where $\epsilon(x) = (-1)^{\sum_{\mu=1}^{d} x_\mu}$

One staggered field gives rise to $2^{D/2}$ spinors in continuum
i.e 2 Dirac in 2D and 4 Dirac in 4D

A mod 2 't Hooft anomaly

Imagine gauging the Z_4 symmetry. Covariant difference operators that leave $\delta S = 0$ under local Z_4 transformations $\chi(x) \to \omega(x)\chi(x)$

$$D_\mu \chi(x) = U_\mu(x)\chi(x+\mu) - U^\dagger_\mu(x-\mu)\chi(x-\mu) \quad U_\mu(x) \in Z_4$$

But measure transforms by factor
$\prod_x \omega(x)^{2N} = \prod_x \left[\omega^2(x)\right]^N = \prod (-1)^N$

$$\boxed{Z \text{ invariant iff } N = 2k}$$

i.e 4k and 8k Dirac spinors in $D = 2, 4$ (8k and 16k Majorana)

> Number of fermions needed to gap boundary states of $2+1$ and $4+1$ topological superconductors without breaking symmetries

Take home

- An exact onsite twisted $U(1)$ chiral symmetry - preserved at the discrete level without fermion doubling.
- Exact lattice anomaly breaks $U(1) \to Z_4$
- Cancelling a 't Hooft anomaly for Z_4 restricts number of fermions in flat space limit to $N_{\mathrm{Weyl}} = 16n$ in $D = 4$.

Symmetric mass generation

Normal (SSB) picture of mass generation:

- Fermion bilinear mass term would break symmetry of system
- If symmetry is unbroken, fermions remain massless
- Spontaneously broken symmetry can form non-zero bilinear condensates leading to massive fermions

SMG = Generation of mass without spontaneous symmetry breaking

One way to generate SMG:

- Four-fermion term explicitly breaks anomalous symmetries but residual symmetry still forbids fermion bilinear condensates
- Massive four fermion condensate can form

Lattice model

$$S = \sum_x \bar{\psi}\,(\eta \cdot \Delta + iy\sigma)\,\psi + \frac{1}{2}\sum_x |\sigma|^2 - \frac{\kappa}{2}\sum_x \sigma\Box\sigma + \frac{\lambda}{4}\sum_x |\sigma|^4$$

ψ: two component staggered fermion field

- $\Delta_\mu \psi = \psi(x+\mu) - \psi(x-\mu)$
- $\eta^\mu(x) = (-1)^{\sum_{i=1}^{\mu-1} x_i}$

σ: scalar in adjoint representation of $SU(2)$

- $|\sigma|^2 = \mathrm{tr}(\sigma^\dagger \sigma)$
- $\Box\sigma = \Delta_\mu^2 \sigma = \sum_x \sigma(x+2\mu) + \sigma(x-2\mu) - 2\sigma(x)$

Weak and strong coupling expansions

Weak coupling:

$$-\mathrm{Tr}\log\left(\eta\cdot\Delta + iy\sigma\right) = \sum_{n=1}(-1)^n\,\mathrm{Tr}\left(iy\;G(x,y)\sigma(y)\right)^n$$

$$S_{\text{eff}} = \frac{y^2}{2}\sum_{x,y}|G(x,y)|^2\,\sigma^a(x)\sigma^b(y), \qquad G = (\eta\cdot\Delta)^{-1}$$

antiferromagnetic ground state

Strong coupling:

$$S_{\text{eff}} = -\sum_x \log\sigma^2(x) + \frac{1}{2y^2|\sigma|^4}\sum_{x,y}|G^{-1}(x,y)|^2\sigma^a(x)\sigma^a(y)$$

non-zero $|\sigma|^2$ but per-site direction (lattice SMG)

Derivative expansion

Expand fluctuations in $\sigma(x)$ around antiferromagnetic background as $\sigma(x) = \mu\varepsilon(x)n^a(x)\tau^a$ with $n^a(x)n^a(x) = 1$.

$$S_{\text{eff}} = -\frac{1}{2}\mathrm{Tr}\ln\left(-\Box + y^2\mu^2 + iy\mu\varepsilon(x)[\eta_\mu(x)\Delta_\mu, n(x)]\right)$$

After some algebraic manipulation and expanding in powers of $\frac{\Delta_\mu}{y\mu}$ the leading term is

$$-\frac{1}{4y^2\mu^2}\mathrm{Tr}(\varepsilon(x)\eta_\mu(x)\Delta_\mu n^a(x)\tau^a)^2 = -\frac{1}{4y^2\mu^2}\sum_x n^a(x)\Box n^a(x)$$

Hopf defects

Next leading order contains a Skyrme term:

$$\frac{1}{16y^4\mu^4}\mathrm{Tr}\left(\epsilon^{abc}\eta_\mu(x)\Delta_\mu(x,y)n^a(y)\eta_\nu(x)\Delta_\nu(x,z)n^b(z)\right)^2 \xrightarrow{\text{continuum}} (\partial_\mu r$$

- Hopf map – non-trivial map between S^2 vacuum manifold arising from constraint $n^a n^a = 1$ and the S^3 spacetime boundary at infinity
- If leading order term is suppressed by tuning κ, marginal Skyrme term dominates – Hopf defects diverge logarithmically with system size.
- Similar to XY model, logarithmic contribution to entropy may cancel large volume supression of action – condensate of defects may be a candidate for continuum SMG

Phase diagram

- an unbroken symmetric phase at small y
- a SMG phase at large y
- a broken antiferromagnetic phase above κ_c
- the three phases meet at a point $P_c = (y_c, \kappa_c)$
- an unexplored frustrated phase for large negative κ

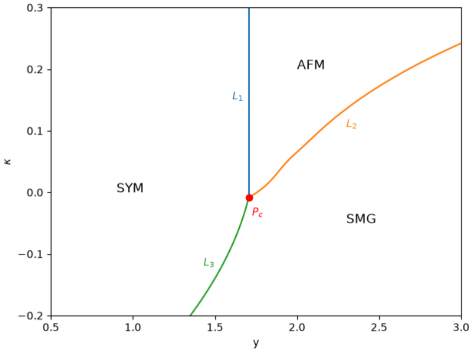


Figure: Phase diagram of the model.

Main observables

$$\phi = \frac{1}{V}\Big|\sum_x \bar{\psi}(x)\psi(x)\Big| \qquad \phi_{\rm stag} = \frac{1}{V}\Big|\sum_x \varepsilon(x)\bar{\psi}(x)\psi(x)\Big|$$

$$\chi = V\big(\langle\phi^2\rangle - \langle\phi\rangle^2\big) \qquad \chi_{\rm stag} = V\big(\langle\phi^2_{\rm stag}\rangle - \langle\phi_{\rm stag}\rangle^2\big)$$

$$\Sigma = \frac{1}{V}\Big|\sum_x \sigma(x)\Big| \qquad \Sigma_{\rm stag} = \frac{1}{V}\Big|\sum_x \varepsilon(x)\sigma(x)\Big|$$

$$\chi_\Sigma = V\big(\langle\Sigma^2\rangle - \langle\Sigma\rangle^2\big) \qquad \chi_{\Sigma,\rm stag} = V\big(\langle\Sigma^2_{\rm stag}\rangle - \langle\Sigma_{\rm stag}\rangle^2\big)$$

Finite size scaling

An infinite system near a continuous phase transition has

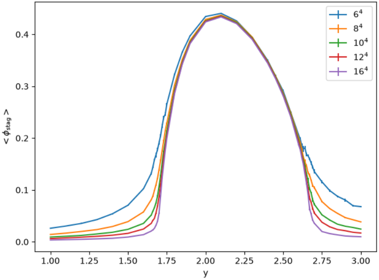
$$\xi \sim |y - y_c|^{-\nu} \qquad \chi \sim |y - y_c|^{-\gamma} \sim \xi^{\gamma/\nu}$$

This implies that $\xi \to \infty$ and $\chi \to \infty$ as $y \to y_c$, but in finite system, maximum correlation length is L!

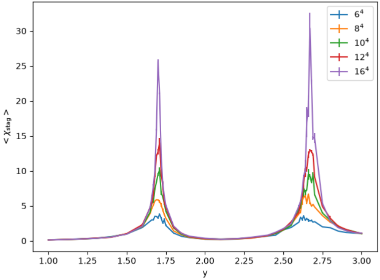
For a finite system we expect

$$\chi_{\text{max}} \sim L^{\gamma/\nu} = L^{2-\eta} \quad \text{and} \quad \xi \sim L \implies |y - y_c| \sim L^{-1/\nu}$$

Can verify these exponents by plotting $\chi / L^{2-\eta}$ vs. $(y - y_c)L^{1/\nu}$ and showing that all values of L give the same curve



(a) $\langle\phi_{\text{stag}}\rangle$ at $\kappa = 0.2$



(b) $\langle\chi_{\text{stag}}\rangle$ at $\kappa = 0.2$

Figure: Anti-ferromagnetic bilinear and its susceptibility vs y at $\kappa = 0.2$

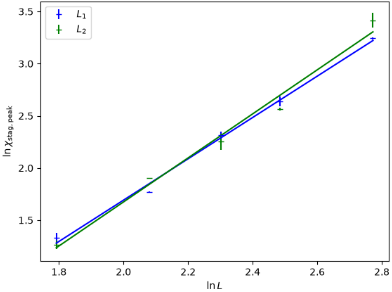


Figure: Peak χ_{stag} versus L for $\kappa = 0.2$ and $\lambda = 1.0$ along the phase transition boundaries L_1 and L_2.

Expect peak susceptibility to scale as a power of L at the phase transition

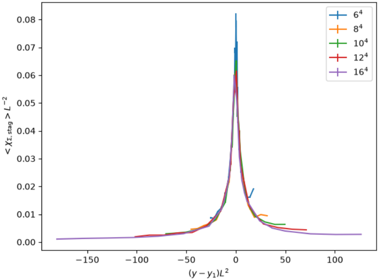
$$\chi_{\text{peak}} \sim L^{2-\eta}$$

Extract η via linear least squares fit of plot $\ln \chi_{\text{peak}}$ versus $\ln L$.

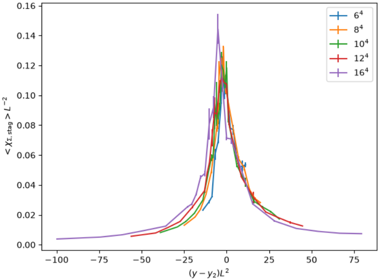
$$\eta_1 = 0.019 \pm 0.077$$

$$\eta_2 = -0.11 \pm 0.15$$

Peak widths



(a) Transition L_1 from symmetric to antiferromagnetic phases



(b) Transition L_2 from antiferromagnetic to SMG phases

Figure: Curve collapse plots for $< \chi_{\Sigma,\mathrm{stag}} >$ for $\kappa = 0.2$ and $\lambda = 1.0$ showing consistency with $\nu = 1/2$

Location of point P_c

Can extrapolate position of merged point via linear fit of transition boundaries close to it:

- $\kappa = 0.025$
- $\kappa = 0.05$
- $\kappa = 0.075$
- $\kappa = 0.1$
- $\kappa = 0.15$

$$y_c = 1.706 \pm 0.003$$
$$\kappa_c = -0.01 \pm 0.006$$

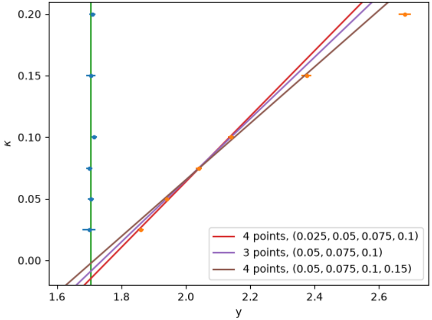
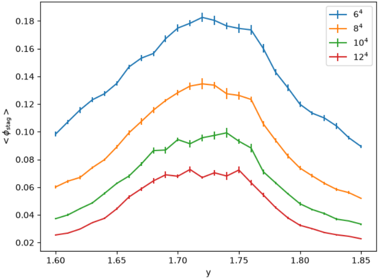
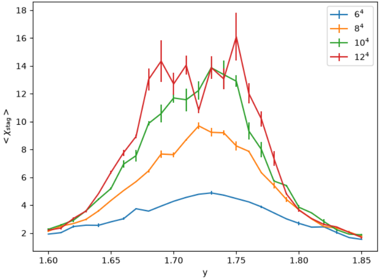


Figure: Location of the two phase transitions and their crossing point



(a) $\langle \phi_{\mathrm{stag}} \rangle$ at $\kappa = -0.01$



(b) $\langle \chi_{\mathrm{stag}} \rangle$ at $\kappa = -0.01$

Figure: Anti-ferromagnetic bilinear and its susceptibility vs y at $\kappa = -0.01$

Critical exponents at P_c

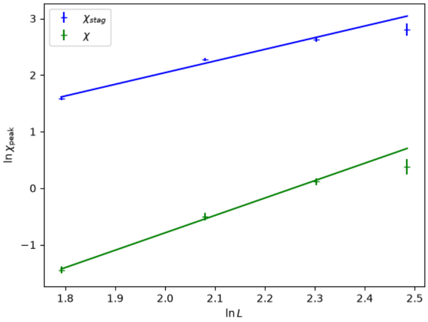


Figure: Peak χ and χ_{stag} versus L for $\kappa = -0.01$ and $\lambda = 1.0$.

Extract η via linear least squares fit of plot $\ln \chi_{\mathrm{peak}}$ versus $\ln L$.

$$\chi_{\mathrm{stag}} : \eta = -0.06 \pm 0.22$$
$$\chi : \eta' = -1.08 \pm 0.15$$

Peak widths and data collapse

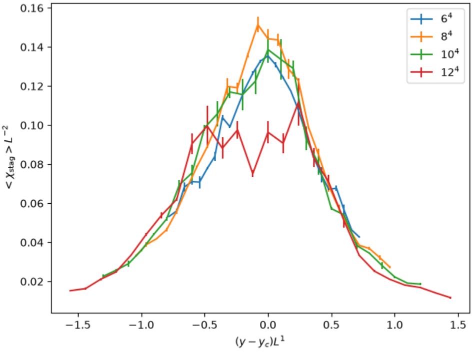
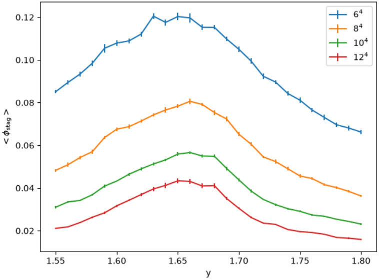


Figure: Curve collapse plot for $< \chi_{\mathrm{stag}} >$ for $\kappa = -0.01$ and $\lambda = 1.0$

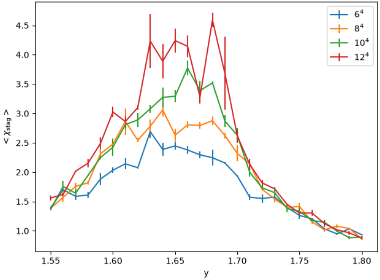
Can collapse the susceptibility data onto a single curve for all volumes by assuming that in the critical region

$$y - y_c \sim \frac{1}{L}.$$

Unusual fixed point not consistent with mean field theory



(a) $< \phi_{\rm stag} >$ vs y at $\kappa = -0.05$



(b) $< \chi_{\rm stag} >$ vs y at $\kappa = -0.05$

Figure: $< \phi_{\rm stag} >$ and $< \chi_{\rm stag} >$ for $\kappa = -0.05$ and $\lambda = 1.0$

Conclusions

- Three phases: free (symmetric) phase, broken antiferromagnetic phase, SMG phase
- Transitions bordering AFM phase consistent with mean field theory
- Candidate multicritical point P_c where phase boundaries merge
- Evidence that a single critical line continues from P_c into negative κ

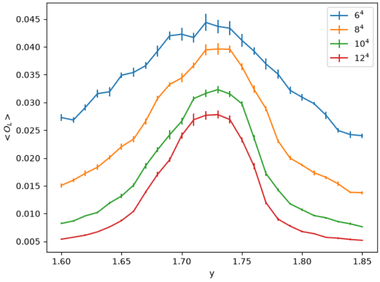
At P_c:

- ferromagnetic and antiferromagnetic order parameters all vanish
- non-zero four fermion condensate
- fermion susceptibilities diverge with non-trivial exponents

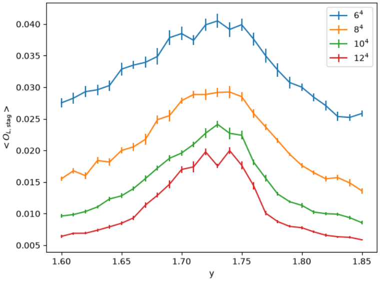
Questions

Extra: One-link operators

$$O_L = \sum_{\mu} \xi_\mu(x)\bar{\chi}(x)\left[\chi(x+\mu) + \chi(x-\mu)\right]$$



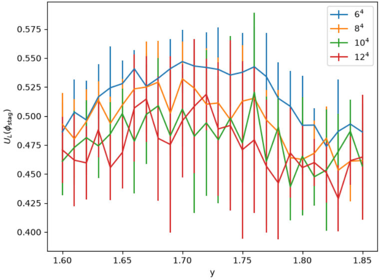
(a) $< O_L >$ vs y at $\kappa = -0.01$ and $\lambda = 1.0$



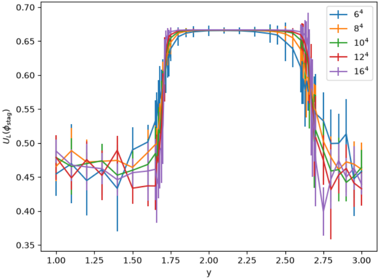
(b) $< O_{L,\mathrm{stag}} >$ vs y at $\kappa = -0.01$ and $\lambda = 1.0$

Extra: Binder cumulant

$$U_L = 1 - \frac{< \phi^4_{\text{stag}} >}{3 < \phi^2_{\text{stag}} >^2}$$



(a) Binder cumulant at $\kappa = -0.01$ and $\lambda = 1.0$



(b) Binder cumulant at $\kappa = 0.2$ and $\lambda = 1.0$